Real Analysis II.

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Assignment XI.

1. **Problem 1.**

   Let $0 \le t < 1$ and let $x_t = (1, t, t^2, ..., t^k, ...,) \in l^2 = H$. Compute the distance between x_{t_0} and $span\{x_{t_1}, ..., x_{t_m}\}$, where $\{t_j\}_{j=0}^m$ are different from each other.

   Hint:
   $$G(x_{t_1}, ..., x_{t_m}) = (t_1 ... t_m)^{-1}\; det\,((t_j^{-1} - t_i)^{-1})_{1 \le i,j \le m}.$$

2. **Problem 2.**

   Let P, Q be projections on closed subspaces M, N of H.

   a) Prove that $Im(PQ) = \{y \in H : PQy = y\} = M \cap N$, provided PQ is a projection.

   b) Prove that PQ is a projection iff $PQ = QP$.

   c) Prove that $P + Q - PQ$ is a projection, provided PQ is a projection.

   d) Prove that $Im(P + Q - PQ) = \{y \in H : (P + Q - PQ)y = y\} = M + N$.

3. **Problem 3.**

   Let M be a closed subspace of H. Prove that for every $x_0 \in H$, we have
   $$min\,\{\|x - x_0\|;\, x \in M\} = max\,\{(x_0, y);\, \|y\| = 1,\, y \in M^{\perp}\}.$$

4. **Problem 4.**

   In this exercise a "projection" is a linear bounded operator $P : H \to H$, satisfying $P^2 = P$.

   a) Prove that ImP, $KerP$ are closed subspaces of H and that $H = ImP + KerP$.

   b) Prove the converse: If $H = M + N$, $M \cap N = \{0\}$, where M, N are closed subspaces of H, then there exists a unique projection P with $M = ImP$, $N = KerP$.

   c) Prove that P is a projection iff $I - P$ is a projection.

   d) Let $a > 0$. Give an example of a projection P satisfying $\|P\| > a$.

   e) Prove that a projection P is orthogonal, ($\|Px\| \le \|x\|$), iff $(Px, y) = (x, Py)$ for all $x, y \in H$.

5. **Problem 5.**

   a) Let $H = C^n$. Assume that the matrix of a linear operator T (with respect to a standard basis) is given as $(a_{ij})_{1 \le i,\, j \le n}$. Prove that $\|T\|^2 \le \sum\limits_{i,j=1}^{n} |a_{i\,j}|^2$.

   b) Let $(a_{ij})_{i,j=1}^{\infty}$ be an infinite Hilbert-Schmidt matrix, $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$. Prove that for every i and for every $x = (x_1, x_2, ...) \in l^2$, $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$ is convergent and $y = (y_1, y_2, ...) \in l^2$. Let $Tx = y$. Prove that T is bounded and $\|T\|^2 \le \sum\limits_{i,j=1}^{\infty} |a_{ij}|^2$.