Real Analysis II.

Instructor: Dmitry Ryabogin

Assignment XII.

1. **Problem 1.**

   Let $e_n = (0, ..., 0, 1, 0, ..., 0, ...)$, (1 stands on nth place), $n = 1, 2, ...$, be a sequence of vectors of l^2. In the case $H = l^2$ give a direct proof of the Riesz theorem by showing that for any linear bounded functional on l^2 we have

   $$\sum_{n=1}^{\infty} |f(e_n)|^2 < \infty. \tag{1}$$

   Hint: Assume that (1) is not true, then there are disjoint subsets E_k of natural numbers, $k = 1, 2, ...$, so that

   $$\sum_{n \in E_k} |f(e_n)|^2 > 1.$$

   Define x_n so that $x_n = c_k f(e_n)$ for $n \in E_k$. For suitably chosen c_k,

   $$\sum_{n=1}^{\infty} x_n f(e_n) = \infty,$$

   although

   $$\sum_{n=1}^{\infty} |x_n|^2 < \infty.$$

2. **Problem 2.**

   Let $\{x_n\}, n = 1, 2, 3, ...$ be a linearly independent system (set) of vectors on H.

   a) Show that the following construction yields an orthonormal system u_n such that $\{x_1, ...x_N\}$ and $\{u_1, ..., u_N\}$ have the same span for all N.

   Hint: Put $u_1 = x_1/\|x_1\|$. Having $u_1, ..., u_{n-1}$ define

   $$v_n = x_n - \sum_{k=1}^{n-1} (x_n, u_k) u_k, \qquad u_n = v_n/\|v_n\|.$$

   Note that this leads to a proof of the existence of a maximal orthonormal system in separable Hilbert spaces.

   b) Let $x_n = x^{n-1}$, $n = 1, 2, 3, ...$ be a linearly independent system of vectors in $L^2(0, 1)$. What are u_n?

   c) Let $x_1 = (1, 0, ..., 0, ...)$, $x_2 = (1, 1/2, 0, ..., 0, ...)$, $x_3 = (1, 0, 1/4, 0, ..., 0, ...)$,..., $x_n = (1, 0, ..., 0, 1/2^{n-1}, 0, ..., 0, ...)$,... be a system of vectors in l^2. Are these vectors independent? Is this system orthonormal? If not, what are u_n? Is it true that the closure of the linear span of $(x_j)_{j=1}^{\infty}$ is l^2? Is $(x_j)_{j=1}^{\infty}$ a basis?

   d) Prove that in $L^2([a, b])$ there are orthonormal systems consisting of polynomials, piecewise linear functions, trigonometric functions.

3. **Problem 3.**

a) If $A \subset [0, 2\pi]$, and A is measurable, prove that

$$\lim_{n\to\infty} \int\limits_A \cos(nx)dx = \lim_{n\to\infty} \int\limits_A \sin(nx)dx = 0.$$

Hint: $1_A \in L^2([0, 2\pi])$.

b) Let $n_1 < n_2 < n_3 < ...$ be positive integers, and let E be the set of all $x \in [0, 2\pi]$ at which $\{\sin(n_k x)\}$ converges. Prove that $\mu(E) = 0$.

Hint: $2\sin^2\alpha = 1 - \cos(2\alpha)$, so $\sin(n_k x) \to \pm 1/\sqrt{2}$ a.e. on E.