1. **Problem 1.**

Let $e_n = (0, ..., 0, 1, 0, ..., 0, ...)$, ($1$ stands on $n$th place), $n = 1, 2, ...$, be a sequence of vectors of $l^2$. In the case $H = l^2$ give a direct proof of the Riesz theorem by showing that for any linear bounded functional on $l^2$ we have

$$
\sum_{n=1}^{\infty} |f(e_n)|^2 < \infty.
$$

(1)

**Hint:** Assume that (1) is not true, then there are disjoint subsets $E_k$ of natural numbers, $k = 1, 2, ...$, so that

$$
\sum_{n \in E_k} |f(e_n)|^2 > 1.
$$

Define $x_n$ so that $x_n = c_k f(e_n)$ for $n \in E_k$. For suitably chosen $c_k$,

$$
\sum_{n=1}^{\infty} x_n f(e_n) = \infty,
$$

although

$$
\sum_{n=1}^{\infty} |x_n|^2 < \infty.
$$

2. **Problem 2.**

Let $\{x_n\}, n = 1, 2, 3, ...$ be a linearly independent system (set) of vectors on $H$.

a) Show that the following construction yields an orthonormal system $u_n$ such that $\{x_1, ..., x_N\}$ and $\{u_1, ..., u_N\}$ have the same span for all $N$.

**Hint:** Put $u_1 = x_1/\|x_1\|$. Having $u_1, ..., u_{n-1}$ define

$$
v_n = x_n - \sum_{k=1}^{n-1} (x_n, u_k) u_k, \quad u_n = v_n/\|v_n\|.
$$

Note that this leads to a proof of the existence of a maximal orthonormal system in separable Hilbert spaces.

b) Let $x_n = x^{n-1}$, $n = 1, 2, 3, ...$ be a linearly independent system of vectors in $L^2(0, 1)$. What are $u_n$?

c) Let $x_1 = (1, 0, ..., 0, ...)$, $x_2 = (1, 1/2, 0, ..., 0, ...)$, $x_3 = (1, 0, 1/4, 0, ..., 0, ...)$,..., $x_n = (1, 0, ..., 0, 1/2^{n-1}, 0, ..., 0, ...)$,... be a system of vectors in $l^2$. Are these vectors independent? Is this system orthonormal? If not, what are $u_n$? Is it true that the closure of the linear span of $(x_j)_{j=1}^{\infty}$ is $l^2$? Is $(x_j)_{j=1}^{\infty}$ a basis?

d) Prove that in $L^2([a, b])$ there are orthonormal systems consisting of polynomials, piecewise linear functions, trigonometric functions.
3. Problem 3.
   a) If $A \subset [0, 2\pi]$, and $A$ is measurable, prove that
   \[
   \lim_{n \to \infty} \int_A \cos(nx)dx = \lim_{n \to \infty} \int_A \sin(nx)dx = 0.
   \]
   **Hint:** $1_A \in L^2([0, 2\pi])$.

   b) Let $n_1 < n_2 < n_3 < \ldots$ be positive integers, and let $E$ be the set of all $x \in [0, 2\pi]$ at which \{\sin(n_kx)\} converges. Prove that $\mu(E) = 0$.
   **Hint:** $2\sin^2 \alpha = 1 - \cos(2\alpha)$, so $\sin(n_kx) \to \pm1/\sqrt{2}$ a.e. on $E$.  