Real Analysis II.

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Assignment II.

1. **Problem 1.** Suppose V is an open set in R^k, T maps V into R^k, and $x \in V$. The map T is called *differentiable* at x if there exists a linear operator A on R^k and a map $\alpha_x : V \to R^k$ such that

$$T(x+h) - T(x) = Ah + \alpha_x(h), \qquad \lim_{|h|\to 0} \frac{|\alpha_x(h)|}{|h|} = 0.$$

Prove that A is unique.

Hint. Observe that

$$\frac{|\alpha_x(\lambda h)|}{|\lambda h|}|h| \to 0,$$

provided $\lambda \in R$, $\lambda \to 0$.

2. **Problem 2.**

a) For $n = 1, 2, 3, ...$ we let P_n be the set of all $x \in R^k$ whose coordinates are integral multiples of 2^{-n}, and we let Ω_n be the collection of all 2^{-n} boxes with corners at points of P_n. Prove that every nonempty *open* set in R^k is a countable union of disjoint boxes belonging to $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup ....$

b) Prove that for any positive translation-invariant Borel measure on R^k such that $\nu(K)$ is finite for every compact set K, there exists a constant $c = c(\nu)$ such that $\nu(E) = c\mu(E)$ for all Borel sets $E \subset R^k$.

Hint. By a) it is enough to check b) for boxes from P_n. Let Q_0 be a 1-box, put $c = \nu(Q_0)$. Then Q_0 is a union of 2^{nk} disjoint 2^{-n} boxes that are translates of each other, and

$$2^{nk}\nu(Q) = \nu(Q_0) = c\mu(Q_0) = c2^{nk}\mu(Q)$$

for every 2^{-n} box.

c) Prove that to every linear transformation T of R^k into R^k corresponds a real number $\Delta(T)$ such that $\mu(T(E)) = \Delta(T)\mu(E)$ for every measurable E. In particular, $\mu(T(E)) = \mu(E)$ when T is rotation.

Hint. Let $m(E) = \mu(T(E))$. Prove that m is translation-invariant and use b).

d) Let T be a linear transformation on R^k defined as follows: $Te_1 = e_1 + e_2$, $Te_i = e_i$ for $i = 2, 3, ..., k$. Assume also that Q is a cube consisting of all $x = (x_1, ..., x_k)$ with $0 \le x_k < 1$. Show that $T(Q)$ is the set of all points $\sum x_i e_i$, whose coordinates satisfy

$$x_1 \le x_2 < x_1 + 1, \qquad 0 \le x_i < 1, \; i \ne 2.$$

Show that $\Delta(T) = |detT|$.

Hint. Let S_1 be the set of points in $T(Q)$ that have $x_2 < 1$ and let S_2 be the rest of $T(Q)$. Observe that $S_1 \cup (S_2 - e_2) = Q$, and that $S_1 \cap (S_2 - e_2)$ is empty. Hence, $\Delta(T) = 1$.

3. **Problem 3.**

   a) Let $f \in L(R)$. Prove that for any $\epsilon > 0$ there exists $g_\epsilon \in C(R)$ supported by a bounded interval, say $[-A, A]$, such that

   $$\int_R |f(x) - g(x)|dx < \epsilon.$$

   b) Let $f \in L(R)$. Prove that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that

   $$\int_R |f(x-t) - f(y-t)|dt < \epsilon,$$

   provided $|x - y| < \delta$.

   c) Prove that $f * g$ is continuous on the real line, provided $f \in L(R)$ and g is bounded.

4. **Problem 4.** Suppose G is a subgroup of R (relative to addition), $G \neq R$, and G is Lebesgue measurable. Prove that $\mu(G) = 0$.

   Hint. Use the last problem of the previous assignment.

5. **Problem** 5***.** Prove that the Cantor function is $\log_3 2$-Lipschitz, i.e., there exists a constant K such that for all $x, y \in [0, 1]$,

   $$|c(x) - c(y)| \leq K|x - y|^{\log_3 2}.$$