Real Analysis II.

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Assignment III.

1. **Problem 1.** Let $T(x,y) = (f(x,y), g(x,y))$ be a map from R^2 to R^2 defined as follows

$$f(x,y) = g(x,y) = \frac{x^2 y}{x^2 + y^2}, \ (x,y) \neq (0,0), \qquad f(0,0) = g(0,0) = 0.$$

   a) Is T continuous at $(0,0)$?

   b) Do partial derivatives $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$ exist?

   c) Is T differentiable at $(0,0)$?

2. **Problem 2. Complex differentiation.**

   A function (a map) $l : \mathbf{C} \to \mathbf{C}$ is called R-linear ($\mathbf{C}$-linear), if

$$l(z_1 + z_2) = l(z_1) + l(z_2) \, \forall z_1, \, z_2 \in \mathbf{C}, \qquad l(\lambda z) = \lambda l(z) \, \forall \lambda \in R, \, \forall z \in \mathbf{C}$$

$$(\, l(z_1 + z_2) = l(z_1) + l(z_2) \, \forall z_1, \, z_2 \in \mathbf{C}, \qquad l(\lambda z) = \lambda l(z) \, \forall \lambda \in \mathbf{C}, \, \forall z \in \mathbf{C}).$$

   a) Prove that any R-linear function is of the form

$$l(z) = az + b\bar{z}, \qquad a = \frac{1}{2}(\alpha - i\beta), \, b = \frac{1}{2}(\alpha + i\beta), \, \alpha = l(1), \, \beta = l(i).$$

   b) Prove that any $\mathbf{C}$-linear function is of the form

$$l(z) = az, \qquad a = l(1).$$

   c) Prove that an R-linear function is $\mathbf{C}$-linear iff $l(iz) = il(z)$.

   d) Compute the Jacobian of an R-linear map and of a $\mathbf{C}$-linear map. What geometric conclusions can you make from the computation?

   e) Let $z \in U \subseteq \mathbf{C}$. A function $f : U \to \mathbf{C}$ is called R-differentiable ($\mathbf{C}$-differentiable) at the point z, provided

$$f(z+h) - f(z) = l(h) + \alpha(z,h), \qquad \lim_{h \to 0} \frac{\alpha(z,h)}{h} = 0,$$

   where l is an R-linear ($\mathbf{C}$-linear) function. Thus,

$$f(z+h) - f(z) = ah + b\bar{h} + \alpha(z,h), \qquad \lim_{h \to 0} \frac{\alpha(z,h)}{h} = 0.$$

   Prove that

$$\frac{\partial f}{\partial x} = a + b, \qquad -i\frac{\partial f}{\partial y} = a - b,$$

or

$$a = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad b = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

f) Complex Analysis starts with the notion of **C**-differentiability. The above formulas show that **C**-differentiability yields $b = 0$. Prove that for $f = u + iv$ the condition $b = 0$ can be written as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the so-called Cauchy-Riemann equations.

g) Observe that $f(z) = x + 2iy$ is nowhere differentiable in the complex sense.

3. **Problem 3. Polar coordinates**.

a) Let $P := \{x \in R^k : x = ((a-t)x_1, (a-t)x_2, ..., (a-t)x_{k-1}, t)\}$, $(x_1, ..., x_{k-1}) \in A$, $t \in [0, a]$ be a pyramid in R^k with a base $A \subset R^{k-1}$ and height a. Prove that

$$vol_k(P) = \frac{a}{k}\, vol_{k-1}(A).$$

Hint. Use Fubini.

b) Let v_k be the volume of the Euclidean ball (of radius 1) in R^k, and let σ_{k-1} be its surface area. Prove that $\sigma_{k-1} = k\, v_k$.

c) Let S^{k-1} be the unit sphere. Show that every $x \in R^k$, except for $x = 0$, has a unique representation of the form $x = ru$, where r is a positive real number and $u \in S^{k-1}$. Thus, R^k may be regarded as a product $(0, \infty) \times S^{k-1}$.

d) Let m_k be Lebesgue measure on R^k, and define a measure σ_{k-1} on S^{k-1} as follows: if $A \subseteq S^{k-1}$ and A is a Borel set, let $\tilde{A}$ be the set of all points ru, where $0 < r < 1$ and $u \in A$, and define $\sigma_{k-1}(A) = k\, m_k(\tilde{A})$. Prove the formula

$$\int_{R^k} f dm_k = \int_0^\infty r^{k-1} dr \int_{S^{k-1}} f(ru) d\sigma_{k-1}(u)$$

for every nonnegative Borel function f on R^k.

Hint. If $0 < r_1 < r_2$ and if A is an open subset of S^{k-1}, let E be the set of all ru with $r_1 < r < r_2$, $u \in A$, and verify the formula for the characteristic function of E. Then approximate.

e) Check that the above formula coincides with familiar results when $k = 2$ and $k = 3$.

f) It is convenient to normalize integrals as above by pulling out the factor kv_k, and write

$$\int_{R^k} f dm_k = kv_k \int_0^\infty r^{k-1} dr \int_{S^{k-1}} f(ru) d\sigma(u), \qquad \sigma(S^{k-1}) = 1.$$

Compute v_k using the following trick: integrate the function $f(x) = \exp\{-\frac{1}{2}\sum_{j=1}^{k} x_j^2\}$ in both ways. This function is at once invariant under rotations and a product of functions depending on separate coordinates. Hence,

$$\int_{R^k} f dm_k = \Pi_{j=1}^k \int_R e^{-x_j^2/2} dx_j = (2\pi)^{k/2} = kv_k \int_0^\infty r^{k-1} e^{-r^2/2} dr = v_k 2^{k/2} \Gamma(k/2+1),$$

and find v_k.

g) It is known that

$$\Gamma\left(\frac{k}{2}+1\right) \approx \sqrt{2\pi} e^{-k/2} \left(\frac{k}{2}\right)^{(k+1)/2}.$$

Conclude that v_k is roughly $(2\pi e/k)^{k/2}$. This is extremely small when k is large. What is going on?