Real Analysis II.

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Assignment IV.

1. **Problem 1.** Make a proper change of variables to compute the double integral

\[ \int_{A} (y^2 - x^2) \, dx \, dy, \quad A := \{(x, y) \in \mathbb{R}^2 : 0 < x < y, xy < 1, x^2 + y^2 < 4\}. \]

2. **Problem 2.**
   a) Let \( K = [-1, 1]^k \) be the cube in \( \mathbb{R}^k \). Compute the volume of \( K \). What is the length of the main diagonal (the segment joining the points \((-1, -1, ..., -1)\) and \((1, 1, ..., 1)\))? 
   b) Prove that the volume of the cube can be written as

\[ \text{vol}_k(K) = v_k \int_{S^{k-1}} r(u)^k \, d\sigma(u), \]

where \( r(u) \) is the "radius" of \( K \) in direction \( u \),

\[ r(u) := \max\{t > 0 : tu \in K\}. \]

Conclude that the "radius" of \( K \) satisfies

\[ \int_{S^{k-1}} r(u)^k \, d\sigma(u) = \frac{2^k}{v_k} \approx \left( \frac{2k}{\pi e} \right)^k, \]

(see 3 g) of the previous assignment), and the "average radius" of \( K \) is about \( \sqrt{2k/\pi e} \). This indicates that the volume of the cube tends to lie in its corners, where the radius close to \( \sqrt{k} \), not in the middle of its facets, where the radius is close to 1.

3. **Problem 2. ”How is the mass of the ball distributed” ?**
   a) Let \( B \) be the ball of volume 1 in \( \mathbb{R}^k \). What is its radius?
   b) The central slice \( A_{e_1}(0) := \{y \in B : y_1 = 0\} \) of \( B \) is an \((k - 1)\)-dimensional ball of the same radius. Prove that the volume of the slice \( \text{vol}_{k-1}(A_{e_1}(0)) = v_{k-1}v_k^{-(k-1)/k} \).
   c) Use

\[ \Gamma\left(\frac{k}{2} + 1\right) \approx \sqrt{2\pi e^{-k/2}} \left(\frac{k}{2}\right)^{(k+1)/2}. \]

\( \Gamma \) to find that \( \text{vol}_{k-1}(A_{e_1}(0)) \approx e \) for large \( k \).
   d) Prove that a parallel slice having a distance \( t \) from the origin

\[ A_{e_1}(t) = \{y \in B : y_1 = t\} \]
has a volume

\[ \text{vol}_{k-1}(A_{e_1}(t)) \approx \sqrt{\frac{e}{r^2}} \left( 1 - \frac{t^2}{r^2} \right)^{(k-1)/2}, \]

where \( r \) is the radius of \( B \).

e) Since \( r \approx \sqrt{k/(2\pi e)} \), prove that

\[ \text{vol}_{k-1}(A_{e_1}(t)) \approx \sqrt{e} e^{-\pi et^2}. \]

f) Draw the graph of \( f(t) := \text{vol}_{k-1}(A_{e_1}(t)) \) and observe that it does not depend on the dimension. Conclude that the volume of the ball concentrates close to \textbf{any} subspace of dimension \( k - 1 \).

g) The part f) suggests that the volume concentrates near the center of the ball, where the subspaces all meet. On the other hand, prove that, for \( k \) large, most of the volume of \( B \) lies near its surface. How do you explain this phenomena?

4. Problem 3.

a) Let

\[ h_\lambda(x) = \frac{2}{\pi} \frac{\lambda}{\lambda^2 + x^2}, \quad \lambda > 0. \]

Prove that

\[ \int_R h_\lambda(x) \, dx = 1 \quad \forall \lambda > 0. \]

b) Let \( g \) be a bounded function. Prove that

\[ \lim_{\lambda \to 0} g * h_\lambda(x) = g(x), \]

provided \( g \) is continuous at \( x \).

c) Let \( g \) be integrable on \( R \). Prove that

\[ \lim_{\lambda \to 0} \int_R |g * h_\lambda(x) - g(x)| \, dx = 0. \]