Real Analysis II.

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Assignment IV.

1. **Problem 1.** Make a proper change of variables to compute the double integral

$$\int_A (y^2 - x^2)dxdy, \qquad A := \{(x, y) \in R^2 : 0 < x < y,\, xy < 1,\, x^2 + y^2 < 4\}.$$

2. **Problem 2.**

   a) Let $K = [-1, 1]^k$ be the cube in R^k. Compute the volume of K. What is the length of the main diagonal (the segment joining the points $(-1, -1, ..., -1)$ and $(1, 1, ..., 1)$)?

   b) Prove that the volume of the cube can be written as

$$vol_k(K) = v_k \int_{S^{k-1}} r(u)^k d\sigma(u),$$

   where $r(u)$ is the "radius" of K in direction u,

$$r(u) := \max\{t > 0 : tu \in K\}.$$

   Conclude that the "radius" of K satisfies

$$\int_{S^{k-1}} r(u)^k d\sigma(u) = \frac{2^k}{v_k} \approx \Big(\frac{2k}{\pi e}\Big)^k,$$

   (use 3 g) of the previous assignment), and the "average radius" of K is about $\sqrt{2k/\pi e}$. This indicates that the volume of the cube tends to lie in its corners, where the radius close to $\sqrt{k}$, not in the middle of its facets, where the radius is close to 1.

3. **Problem 2. "How is the mass of the ball distributed" ?**

   a) Let B be the ball of volume 1 in R^k. What is its radius?

   b) The central slice $A_{e_1}(0) := \{y \in B : y_1 = 0\}$ of B is an $(k-1)$-dimensional ball of the same radius. Prove that the volume of the slice $vol_{k-1}(A_{e_1}(0)) = v_{k-1}v_k^{-(k-1)/k}$.

   c) Use

$$\Gamma\Big(\frac{k}{2} + 1\Big) \approx \sqrt{2\pi}e^{-k/2}\Big(\frac{k}{2}\Big)^{(k+1)/2}.$$

   to find that $vol_{k-1}(A_{e_1}(0)) \approx e$ for large k.

   d) Prove that a parallel slice having a distance t from the origin

$$A_{e_1}(t) = \{y \in B : y_1 = t\}$$

has a volume

$$vol_{k-1}(A_{e_1}(t)) \approx \sqrt{e}\Big(1 - \frac{t^2}{r^2}\Big)^{(k-1)/2},$$

where r is the radius of B.

e) Since $r \approx \sqrt{k/(2\pi e)}$, prove that

$$vol_{k-1}(A_{e_1}(t)) \approx \sqrt{e}\ e^{-\pi e t^2}.$$

f) Draw the graph of $f(t) := vol_{k-1}(A_{e_1}(t))$ and observe that it does not depend on the dimension. Conclude that the volume of the ball concentrates close to **any** subspace of dimension $k-1$.

g) The part f) suggests that the volume concentrates near the center of the ball, where the subspaces all meet. On the other hand, prove that, for k large, most of the volume of B lies near its surface. How do you explain this phenomena?

4. **Problem 3.**

a) Let

$$h_\lambda(x) = \frac{2}{\pi}\frac{\lambda}{\lambda^2 + x^2}, \qquad \lambda > 0.$$

Prove that

$$\int_R h_\lambda(x)dx = 1 \qquad \forall \lambda > 0.$$

b) Let g be a bounded function. Prove that

$$\lim_{\lambda \to 0} g * h_\lambda(x) = g(x),$$

provided g is continuous at x.

c) Let g be integrable on R. Prove that

$$\lim_{\lambda \to 0} \int_R |g * h_\lambda(x) - g(x)|dx = 0.$$