Real Analysis.

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Assignment V.

1. **Problem 1.**

   a) Let $B^n_\infty := [-1,1]^n$ be a cube in R^n. Prove that for $x = (x_1, x_2, ..., x_n) \in R^n$, $\|x\|_{B^n_\infty} = max_{1\le j\le n}|x_j|$.

   b) Let $B^n_1 := convhull\,\{(\pm 1, 0, ...,), (0, \pm 1, 0, ..., 0), ..., (0, 0, ..., 0, \pm 1)\}$ be an octahedron in R^n. Prove that for $x = (x_1, x_2, ..., x_n) \in R^n$, $\|x\|_{B^n_1} = \sum\limits_{j=1}^n |x_j|$.

   c) Let $\|x\|_{B^n_p} := \left(\sum\limits_{j=1}^n |x_j|^p\right)^{1/p}$, $0 < p < \infty$. Prove that $\lim\limits_{p\to\infty} \|x\|_{B^n_p} = \|x\|_{B^n_\infty}$.

2. **Problem 2.**

   a) Let $f(x) = x^\alpha$, $x \in (0,1]$, $\alpha \in \mathbf{R}$. Describe α for which $f \in L^p(m, [0,1])$ for (fixed) $p \ge 1$.

   b) Let $\mu(X) = +\infty$, and let $p \ge 1$. Construct a function $f \in L^p(\mu, X)$ such that $f \notin L^r(\mu, X)$ for $r \ge 1$, $r \ne p$.

   c) Let $\mu(X) = +\infty$, and let $1 \le s < p < \infty$. Prove that $f \in L^s(\mu, X)$, $f \in L^p(\mu, X)$ implies $f \in L^r(\mu, X)$ for $s < r < p$.

   d) Let $\mu(X) = +\infty$. Construct a function $f \notin L^\infty(\mu, X)$, but $f \in L^p(\mu, X)$ $\forall p \ge 1$.

   e) Let $\|f\|_p := \left(\int\limits_X |f|^p d\mu\right)^{1/p}$, $\|f\|_\infty := \inf_{\{A\subset X:\,\mu(A)=0\}} \sup_{\{x\in X\setminus A\}} |f(x)|$. Prove that $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$, provided $\|f\|_r < \infty$ for some $r < \infty$.

3. **Problem 3.** A real function ϕ defined on an open set (a,b), $-\infty \le a < b \le \infty$ is called *convex* if the inequality $\phi((1-\lambda)x + \lambda y) \le (1-\lambda)\phi(x) + \lambda\phi(y)$ holds whenever $a < x < b$, $a < y < b$, and $0 \le \lambda \le 1$.

   a) Prove that convexity of ϕ is equivalent to the requirement that

   $$\frac{\phi(t) - \phi(s)}{t - s} \le \frac{\phi(u) - \phi(t)}{u - t}, \qquad a < s < t < u < b.$$

   b^*) Prove that convex function on (a,b) is continuous on (a,b).

   Hint. See Rudin, page 61.

   c) Prove that the supremum of any collection of convex functions on (a,b) is convex (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

d) If ϕ is convex on (a, b), and if ψ is convex and nondecreasing on the range of ϕ, prove that $\psi(\phi)$ is convex on (a, b). For $\phi > 0$ show that the convexity of $\log \phi$ implies the convexity of ϕ, but not vice versa.

e) Let f be an increasing function on $[a, b]$. Prove that $F(x) := \int\limits_a^x f(t)dt$ satisfies $F((x + y)/2) \leq (F(x) + F(y))/2$.

f) Let ϕ be continuous on (a, b) and let $\phi((x + y)/2) \leq (\phi(x) + \phi(y))/2$. Prove that

$$\phi\Big(\frac{x_1 + x_2 + ... + x_n}{n}\Big) \leq \frac{\phi(x_1) + \phi(x_2) + ... + \phi(x_n)}{n}.$$

Hint. Prove the inequality for $n = 2^m$. For $n \neq 2^m$ use the Cauchy trick: choose $m : 2^m > n$ and look at

$$A := \frac{x_1 + x_2 + ... + x_n}{n}, \qquad A = \frac{(x_1 + x_2 + ... + x_n) + (2^m - n)A}{2^m}.$$

g) Let ϕ be convex on (a, b). Prove that ϕ satisfies the **Jensen inequality**,

$$\phi(\sum_{j=1}^{n} \lambda_j x_j) \leq \sum_{j=1}^{n} \lambda_j \phi(x_j), \qquad \forall x_j \in (a, b),\ \lambda_j \in [0, 1],\ \sum_{j=1}^{n} \lambda_j = 1.$$

Hint. Prove at first that $\sum\limits_{j=1}^{n} \lambda_j x_j \in (a, b)$, consider $(a, b) = (0, 1)$ first. Then use induction.

h)* Prove that if f is **continuous** and satisfies $f((x+y)/2) \leq (f(x)+f(y))/2$, $\forall x, y \in (a, b)$, then f is convex.

Hint. Use g) for **rational** λ_j. Then pass to the limit.