Real Analysis.

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Assignment VI.

1. **Problem 1.**

   a) Let K be a convex origin-symmetric set in R^n. Define the **polar body** K^* of K as $K^* := \{x \in \mathbf{R}^n : x \cdot y \le 1 \ \forall y \in K\}$. Prove that $(K^*)^* = K$.

   Hint. To show that $(K^*)^* \subseteq K$ assume the contrary: there exists $x \in (K^*)^* \setminus K$. Prove that there exists a hyperplane $H_{\alpha,u} := \{y \in R^n : y \cdot u = \alpha\}$, such that K lies in a half-space $\{y \in R^n : y \cdot u \le \alpha\}$, and $x \in \{y \in R^n : y \cdot u > \alpha\}$. What can you say about $z = u/\alpha$?

   b) Let f be a differentiable function in R^n, homogeneous of degree m, that is $g(tx) = t^m g(x)$ for every $t > 0$. Prove Euler's formula: $mg(x) = \nabla g(x) \cdot x$.

   c) Assume that a convex body K is such that its support function h_K is differentiable in $R^n \setminus \{0\}$. Prove that $h_K(x) = \nabla h_K(x) \cdot x$. Conclude that $\nabla h_K(x)$ belongs to a tangent plane $H := \{y \in R^n : y \cdot x = h_K(x)\}$ to K at the point $\nabla h_K(x)$.

   d) Use convexity of K to prove that for every $x, y \in K^*$, $\nabla h_K(x) \cdot x \ge \nabla h_K(x) \cdot y$.

   e) Use $h_K(x) = \nabla h_K(x) \cdot x$ of c) and a) to prove $\nabla h_K(x) \in (K^*)^* = K$. Conclude that $\nabla h_K(x) \in K \cap H$.

2. **Problem 2.**

   a) Let K be a convex origin-symmetric body in R^n, $\rho_K(x) := \max\{\lambda > 0 : \lambda x \in K\}$ be the distance function of K and let $A \in GL(n)$ (an invertible linear transformation from R^n to R^n). Prove that $\rho_{AK}(x) = \rho_K(A^{-1}x)$.

   b) Let A, K be as in a). Prove that $h_{AK}(x) = h_K(A^t x)$, where A^t is a conjugate operator, defined as $Ax \cdot y = x \cdot A^t y$.

   c) Use $\rho_{K^*}(u) = 1/h_K(u)$, $u \in S^{n-1}$, to show that $h_{(AK)^*}(u) = h_{(A^{-1})^t K^*}(u)$. Conclude that $(AK)^* = (A^{-1})^t K^*$.

   d) Conclude that for any $A \in GL(n)$, $vol_n(AK)vol_n((AK)^*) = vol_n(K)vol_n(K^*)$.

   e)***

   Prove that for any K,

   $$vol_n(K)vol_n(K^*) \ge vol_n(B_1^n)vol_n(B_\infty^n) = \frac{4^n}{n!}$$

   .

   Hint. I have no clue.

3. **Problem 3.**

a) Let K, L be two convex origin-symmetric bodies in R^n. Prove that

$$(K \cap L)^* = \cup_{\theta \in [0,1]}((1-\theta)K^* + \theta L^*).$$

b) Prove that $(B_n^2)^* = B_n^2$.

c) Let $K := \{x \in \mathbf{R}^n : \sum_{k=1}^{n} |x_k|^2 / a_k^2 \leq 1,\ a_k > 0\}$ be an ellipsoid. Find K^*.

d) Let $1 \leq p < \infty$, and let $B_n^p := \{x \in \mathbf{R}^n : \sum_{k=1}^{n} |x_k|^p \leq 1\}$, $B_n^\infty := \{x \in \mathbf{R}^n : \sup_{k=1,\dots,n} |x_k| \leq 1\}$.

Prove that $(B_n^p)^* = B_n^q$, where $1/p + 1/q = 1$.

Hint. You might want to use the Lagrange multipliers. Then define

$$F_x(y) := x \cdot y - \lambda(\sum_{k=1}^{n} y_k^p - 1), \qquad \lambda > 0.$$

Then (we might assume that $y_i, x_i > 0$, we also assume that y is precisely the point on the boundary where the max is achieved)

$$\frac{\partial F}{\partial y_i} = 0,\ \frac{\partial F}{\partial \lambda} = 0, or\ x_i = \lambda p y_i^{p-1},\ \sum_{k=1}^{n} y_k^p = 1.$$

Rising the first equation to the power $p/(p-1)$ and summing up in i, we see that the second condition implies

$$\sum_{i=1}^{n} x_i^{p/(p-1)} = (\lambda p)^{p/(p-1)}, \qquad \Big(\sum_{i=1}^{n} x_i^{p/(p-1)}\Big)^{(p-1)/p} = \lambda p.$$

But,

$$\sum_{i=1}^{n} y_i x_i = \lambda p \sum_{i=1}^{n} y_i^p = \lambda p,$$

and

$$h_{B_n^p}(x) = x \cdot y = \Big(\sum_{i=1}^{n} x_i^{p/(p-1)}\Big)^{(p-1)/p}, \qquad x \in S^{n-1}.$$

Observe that we have a maximum:

$$\frac{\partial^2 F}{\partial y_i \partial y_j} = 0, \qquad j \neq i, \qquad \frac{\partial^2 F}{\partial y_i^2} = -\mu p(p-1) y_i^{p-2} < 0.$$