Real Analysis.

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Assignment VII.

1. **Problem 1.**

   a) Let $\mu(X) = 1$, and suppose f and g are positive measurable functions on X such that $fg \geq 1$. Prove that

   $$\int_X f d\mu \int_X g d\mu \geq 1.$$

   b) Let X and μ be as above, and let $h \geq 0$ be measurable. If $A = \int_X h d\mu$, prove that

   $$\sqrt{1 + A^2} \leq \int_X \sqrt{1 + h^2} d\mu \leq 1 + A.$$

2. **Problem 2.**

   a) Suppose μ is a positive measure on X and a positive f satisfies $\int_X f d\mu = 1$. Prove that, for every $E \subset X$ with $0 < \mu(E) < \infty$, that

   $$\int_E log f d\mu \leq \mu(E) log \frac{1}{\mu(E)}.$$

   b) If f is a positive measurable function on $[0, 1]$, which is larger,

   $$\int_0^1 f(x) log f(x) dx, \qquad or \qquad \int_0^1 f(x) dx \int_0^1 log f(x) dx\,?$$

3. **Problem 3.** Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to the Lebesgue measure, and

   $$F(x) := \frac{1}{x} \int_0^x f(t) dt, \qquad x \in (0, \infty).$$

   a) Prove the Hardy inequality

   $$\|F\|_p \leq \frac{p}{p-1} \|f\|_p\,,$$

   which is that the mapping $f \to F$ carries L^p into L^p.

Hint. Write

$$xF(x) = \int_0^x f(t)t^{\alpha}t^{-\alpha}dt, \qquad 0 < \alpha < 1/q,$$

use Holder's inequality to get an upper bound for $F^p(x)$, and integrate to obtain

$$\int_0^\infty F^p(x)dx \le (1-\alpha q)^{1-p}(\alpha p)^{-1} \int_0^\infty f^p(t)dt.$$

b) Show that the best choice of α yields

$$\int_0^\infty F^p(x)dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(t)dt.$$

c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

Hint. Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A.

d) If $f > 0$, and $f \in L^1$, prove that $F \notin L^1$.

e) Prove that if

$$\int_0^\infty f(t)dt = 0, \qquad \int_0^\infty |f(t)log|t||dt < \infty,$$

then $F \in L^1$.

4. **Problem 4.** Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu, X)$, $\|f\|_\infty > 0$, and $\alpha_n := \int_X |f(x)|^n d\mu$, $n \in \mathbf{N}$. Prove that $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = \|f\|_\infty$.

   Hint.

   a) Prove that for **any** sequence of positive numbers $(\alpha_n)_{n=1}^\infty$, $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = \lim_{n\to\infty} \alpha_n^{1/n}$, provided the first limit exists.

   b) Using Holder's inequality prove that the limit $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n$, $\alpha_n := \int_X |f(x)|^n d\mu$, exists.

   c) Use $\|f\|_n \to \|f\|_\infty$ as $n \to \infty$ and a), b).