Real Analysis.

Instructor: Dmitry Ryabogin

Assignment VIII.

1. **Problem 1.**

   a) Let $1/p+1/q=1$, $f \in L^p(R^n)$, $g \in L^q(R^n)$, and $h = f * g$, prove that h is uniformly continuous on R^n. If also $1 < p < \infty$, then $h(x) \to 0$ as $|x| \to \infty$; show that this fails for some $f \in L^1(R^n)$, $g \in L^\infty$.

   b) Suppose $1 \le p < \infty$, $f \in L^p(R)$, and

   $$g(x) = \int_x^{x+1} f(t)dt.$$

   Prove that $g(x) \to 0$ as $|x| \to \infty$. What can you say about g if $f \in L^\infty(R)$?

2. **Problem 2.**

   a) Discover the conditions on f, g for which the equality happens in Minkowski's inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

   Hint. If you want you can use Rudin's book.

   b) If $1 < p < \infty$, prove that the unit ball of $L^p(\mu)$ is *strictly convex*; this means that if

   $$\|f\|_p = \|g\|_p = 1, \qquad f \ne g, \qquad h = \frac{1}{2}(f+g),$$

   then $\|h\|_p < 1$. (Geometrically, the surface of the ball contains no straight lines). Show that this fails in every $L^1(\mu)$, and in every $L^\infty(\mu)$.

3. **Problem 3.**

   a) Let (X, Ω, μ) be a (positive) measure space, and let A_0 be the collection of all sets $E \in \Omega$ with *positive* measure. Prove that $L^p(\mu) \subset L^q(\mu)$ for some p, q, $1 \le p < q$ iff $\inf_{\{E \in A_0\}} \mu(E) > 0$.

   Hint. Prove that there exists a constant $k > 0$ such that for every $E \in \Omega$,

   $$\mu(E)^{1/q} \le k\mu(E)^{1/p}.$$

   To show another direction define $E_n = \{x \in X : |f(x)| > n\}$, $n = 1, 2, ...$ and show that $\mu(E_n) \to 0$ as $n \to \infty$. Hence, there exists n_0 such that $\mu(E_n) = 0$ for $n \ge n_0$, and $f \in L^\infty(\mu)$.

   b) Let (X, Ω, μ) be as above, and let A_∞ be the collection of all sets $E \in \Omega$ with *finite* measure. Prove that $L^q(\mu) \subset L^p(\mu)$ for some $1 \le p < q$ iff $\sup_{\{E \in A_\infty\}} \mu(E) < \infty$.

   Hint. Prove that there exists a constant $k > 0$ such that for every $E \in \Omega$,

   $$\mu(E)^{1/p} \le k\mu(E)^{1/q}.$$

To show another direction define $F_n = \{x \in X : 1/(n+1) \leq f(x) < 1/n\}$, $n = 1, 2, ...$ Then

$$\mu(F_n) \leq (n+1)^q \int_X |f|^q d\mu \leq C < \infty, \qquad , n = 1, 2, ...,$$

yields $\sum_{n=1}^{\infty} \mu(F_n) < \infty$. Now use $p < q$, and the standard decomposition of X into $\{x \in X : |f(x)| \leq 1\}$, and $\{x \in X : |f(x)| > 1\}$.

4. **Problem 4.** Place balls of radius $1/2$ into each of the 2^n vertices of the unit cube $[0, 1]^n$ so that they touch along the edges of the cube, and consider the ball concentric with cube and just touching the other balls. This ball is quite small, right? Is it really contained in the cube for all n?