Real Analysis.

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Assignment IX.

1. **Problem 1.**

   a) Show that the vectors $(x_j)_{j=1}^n$ in H are linearly independent, provided they are pairwise orthogonal.

   b) Prove that $x = 0$ iff $(x, y) = 0$ for all $y \in H$.

2. **Problem 2.**

   Let H be a space of complex polynomials of degree $\le n$.

   a) Show that the dimension of H is $n + 1$.

   b) Let $\lambda_0 < \lambda_1 < ... < \lambda_n$ be constants. Define

   $$(P, Q) = \sum_{j=0}^{n} P(\lambda_j)\overline{Q(\lambda_j)}.$$

   Prove that this is a scalar product, and prove that H is a Hilbert space with respect to it.

3. **Problem 3. Matrices of Hilbert-Schmidt**.

   Let M be a collection of all infinite matrices over complex numbers, $\mathbf{a} = ((a(i,j))_{1\le i,j<\infty}$, such that $a(i,j) = 0$ for $n(\mathbf{a}) = n < max\{i,j\}$, (all terms $a(i,j)$ are zero except for finitely many terms).

   a) Prove that M is a linear space with respect to usual operations, and that one can define a multiplication in M.

   Define a **trace** of $\mathbf{a} \in M$ as follows

   $$tr\,\mathbf{a} = \sum_i a(i,i).$$

   b) Let α, β be complex numbers, and let $\mathbf{a}, \mathbf{b} \in M$. Prove that

   $$tr\,\mathbf{ab} = tr\,\mathbf{ba}; \qquad tr\,\mathbf{a}^*\mathbf{a} \ge 0; \qquad tr\,(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha tr\,\mathbf{a} + \beta tr\,\mathbf{b}.$$

   Here $\mathbf{a}^* = (\overline{(a(j,i)})_{1\le i,j<\infty}$.

   For $\mathbf{a}, \mathbf{b} \in M$ define

   $$(\mathbf{a}, \mathbf{b}) = tr\,\mathbf{ab}^*.$$

   c) Prove that this is a scalar product, and that

   $$(\mathbf{a}, \mathbf{b}) = \sum_{ij} a(i,j)\overline{b(i,j)}.$$

d*) Show that, if we complete M with respect to this scalar product, we will get a collection of infinite matrices for which

$$\sum_{ij} |a(i,j)|^2 < \infty.$$

4. **Problem 4*.** Prove that the **algebraic dimension** of l^2 is $2^{\aleph_0}$.

   Hint: To get a lower bound look at a collection $(x_t)_{0 \le t < 1}$, $x_t = (1, t, t^2, t^3, ..., t^k, ...)$, and use the **determinant of Vandermonde**.

5. **Problem 5.** Give a precise formula for the projection $P_C(x) := dist(x, C)$.

   a) $C = B(z, r)$,

   b)
   $$C = \{x \in l^2 : x_{i_k} = \lambda_k, \; k = 1, 2, 3, ...\}, \qquad (\lambda_k)_{k=1}^{\infty} \in l^2,$$
   where i_k is an increasing sequence of of natural numbers.

   c) C is a collection of all real non-negative (up to a set of measure zero) functions in $L^2([0,1])$.