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An analogue of polynomially integrable bodies in even-dimensional spaces ☆

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ABSTRACT

A bounded domain $K \subset \mathbb{R}^n$ is called polynomially integrable if the $(n-1)$-dimensional volume of the intersection K with a hyperplane Π polynomially depends on the distance from Π to the origin. It was proved in [7] that there are no such domains with smooth boundary if n is even, and if n is odd then the only polynomially integrable domains with smooth boundary are ellipsoids. In this article, we modify the notion of polynomial integrability for even n and consider bodies for which the sectional volume function is a polynomial up to a factor which is the square root of a quadratic polynomial, or, equivalently, the Hilbert transform of this function is a polynomial. We prove that ellipsoids in even dimensions are the only convex infinitely smooth bodies satisfying this property.

1. Formulation of the problem and the main result

The following notion was introduced in [1].

Definition 1.1. Let K be a bounded domain in $\mathbb{R}^n$. Then K is called *polynomially integrable* if the Radon transform of its characteristic function

$$A_K(\xi,t) = R\chi_K(\xi,t) = \int\limits_{K\cap\{x\cdot\xi=t\}} dx,\ \xi\in S^{n-1},\ t\in\mathbb{R},$$

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is a polynomial in t:

$$A_K(\xi,t)=\sum_{j=0}^{N} a_j(\xi)t^j$$

for t such that the hyperplane $x\cdot\xi=t$ intersects K.

Polynomially integrable domains with C^∞ boundary were fully characterized in [1], [7]. First, there are no such domains in $\mathbb{R}^n$ with even n. Secondly, if n is odd then ellipsoidal domains exhaust the class of such domains:

Theorem 1.2 *([7]). Let K be a bounded domain in $\mathbb{R}^n$ with an infinitely smooth boundary ∂K. If K is polynomially integrable then n is odd and K is an ellipsoid.*

Remark 1.3. Theorem 1.2 was formulated in [7] for convex bodies K. However, it was proved in [1] that polynomially integrable domains in $\mathbb{R}^{2k+1}$, with smooth boundary, are necessarily convex and thus the convexity assumption in Theorem 1.2 is superfluous. Also, when K is a convex body, the function $A_K(\xi,t)$ is continuous with respect to ξ, which implies that the coefficients $a_j(\xi)$ are a priori continuous functions on the unit sphere.

In this article, we introduce an analogue of polynomial integrability in even dimensions. First of all, there are no polynomially integrable convex domains with smooth boundary in even-dimensional spaces. This was proved in [1] and [7] using different arguments. The proof in [1] relies on the behavior of the sectional volume function $A_K(\xi,t)$ near the tangent plane $T_a(\partial K)=\{x\cdot\xi=t_0\}$ to the boundary at a point $a\in\partial K$ (see Lemma 2.2). The argument is as follows: for almost all normal vectors $\xi\in S^{n-1}$ near the tangent plane we have $A_K(\xi,t)=const\ (t-t_0)^{\frac{n-1}{2}}(1+o(1)),\ t\to t_0$. If n is even then $\frac{n-1}{2}$ is half-integer and therefore $A_K(\xi,t)$ cannot be a polynomial in t.

In order to formulate the main result of the article we need some notations. The support functions of a compact convex body $K\subset\mathbb{R}^n$ are defined by

$$h_K^+(\xi)=h_K(\xi)=\max_{x\in K} x\cdot\xi, \tag{1}$$

$$h_K^-(\xi)=\min_{x\in K} x\cdot\xi, \tag{2}$$

where ξ belongs to the unit sphere S^{n-1} in $\mathbb{R}^n$. Clearly, $h_K^-(\xi)=-h_K^+(-\xi)$ and a hyperplane $\{x\cdot\xi=t\}$ meets the interior of K if and only if $t\in I_\xi:=\big(h_K^-(\xi),h_K^+(\xi)\big)$.

Denote by $\mathcal{H}$ the Hilbert transform

$$\mathcal{H}f(t)=\frac{1}{\pi}p.v.\int_{\mathbb{R}}\frac{f(s)}{t-s}ds \tag{3}$$

of a continuous function f with sufficiently fast decay at infinity.

The main result of this article is as follows.

Theorem 1.4. *Let n be an even positive integer. Let K be a bounded convex domain in $\mathbb{R}^n$ with C^∞ boundary ∂K. The following are equivalent:*

(i) The sectional volume function $A_K(\xi,t)$ has for $t\in I_\xi$ the form

$$A_K(\xi,t)=\sqrt{q(\xi,t)}P(\xi,t),$$

where $P(\xi,t)$, $q(\xi,t)$ *are continuous in* ξ *and polynomials in* t *with* $\deg q(\xi,\cdot)=2$; $q(\xi,t)>0, t\in I_\xi$.

(ii) The sectional volume function $A(\xi,t)$ *has for* $t\in I_\xi$ *the form*

$$A_K(\xi,t)=\frac{P(\xi,t)}{\sqrt{q(\xi,t)}},$$

where $P(\xi,t)$, $q(\xi,t)$ *are as in (i).*

(iii) The Hilbert transform $\mathcal{H}A_K(\xi,t)$ *is a polynomial with respect to* $t\in I_\xi$ *for each* $\xi\in S^{n-1}$, *i.e.,*

$$\mathcal{H}A_K(\xi,t)=\sum_{j=0}^{N} b_j(\xi)t^j,$$

where N *is an integer and* b_j *are some (a priori continuous) functions on the unit sphere.*

(iv) K *is an ellipsoid.*

2. Proof of Theorem 1.4. Equivalence of conditions (i), (ii), (iii)

We start with some preliminary facts.

2.1. Boundary behavior of the sectional volume function

In the case where K is an ellipsoid, the support function $h_K(\xi)$ is the restriction to the unit sphere $|\xi|=1$ of the square root of a quadratic polynomial. In fact, for the ellipsoid E written in suitable coordinates in the standard form

$$E=\left\{\sum_{j=1}^{n}\frac{x_j^2}{a_j^2}\le 1\right\}$$

we have

$$h_E(\xi)=\sqrt{\sum_{j=1}^{n}a_j^2\xi_j^2}.$$

Also one can check that

$$A_E(\xi,t)=C_n\mathrm{Vol}_n(E)\,h_E^{-n}(\xi)\left(h_E^2(\xi)-t^2\right)^{(n-1)/2}, \tag{4}$$

for a certain constant C_n, and all ξ and t such that $x\cdot\xi=t$ intersects E. It follows that if n is odd then $A_K(\xi,t)$ is a polynomial in t and if n is even then $A_K(\xi,t)$ has the form (ii) in Theorem 1.4 with $q(\xi,t)=h_E^2(\xi)-t^2$.

A hyperplane $\{x\cdot\xi=t\}$ meets the domain K if and only if $t\in I_\xi=[h_K^-(\xi),\ h_K^+(\xi)]$ and the end points $t=h_K^\pm(\xi)$ of the segment I_ξ correspond to the tangent hyperplanes

$$T_{a^\pm}(\partial K)=\{x\cdot\xi=h_K^\pm(\xi)\}$$

at the points $a^\pm\in\partial K$ such that the exterior unit normal vectors $\nu_{\partial K}(a^\pm)$ are correspondingly $\nu_{\partial K}(a^\pm)=\pm\xi$.

The behavior of the sectional volume function $A_K(\xi,t)$ near the tangent planes is given by the following Lemma (see [3, Ch. 1, Section 1.7], [1, Section 3, p. 7], [2, Lemma 2.2]).

Lemma 2.1. *There is a dense subset $\Sigma \subset S^{n-1}$, such that the following asymptotic relation with respect to t holds with some nonzero coefficients $c^{\pm}(\xi)$, non-vanishing for $\xi \in \pm\Sigma$, correspondingly:*

$$A_K(\xi,t) = c^+(\xi)(h_K^+(\xi) - t)^{\frac{n-1}{2}}(1+o(1)),\ t \to h_K^+(\xi) - 0,\ \xi \in \Sigma, \tag{5}$$

$$A_K(\xi,t) = c^-(\xi)(t - h_K^-(\xi))^{\frac{n-1}{2}}(1+o(1)),\ t \to h_K^-(\xi) + 0,\ \xi \in -\Sigma. \tag{6}$$

Proof. We will use the notation $\Gamma = \partial K$. Then Γ is an infinitely differentiable closed hypersurface. Let $\kappa_\Gamma(a),\ a \in \Gamma$ be the Gaussian curvature of Γ at the point a.

Denote by γ the Gauss mapping

$$\gamma : \Gamma \ni a \to \nu_\Gamma(a) \in S^{n-1},$$

which maps a point $a \in \Gamma$ to the exterior unit normal vector $\gamma(a) = \nu_\Gamma(a)$ to Γ at the point a. The mapping γ is differentiable and the Gaussian curvature $\kappa_\Gamma(a)$ is equal to the Jacobian determinant $\kappa_\Gamma(a) = J_\gamma(a)$ of γ at the point a. Therefore, the points a with $\kappa_\gamma(a) \neq 0$ (*non-degenerate points*) constitute the set Reg_γ of regular points of the mapping γ, while the set of points a of zero Gaussian curvature coincides with the critical set Crit_γ.

By Sard's theorem (see e.g., [8, Sections 2 and 3]), the set $\gamma(\mathrm{Crit}_\gamma)$ has the Lebesgue measure zero on S^{n-1}, while the set

$$\Sigma = S^{n-1} \setminus \gamma(\mathrm{Crit}_\gamma)$$

of regular values is a dense subset of S^{n-1}. It consists of the unit vectors ξ such that any point $a \in \Gamma$ with $\nu_\Gamma(a) = \xi$ is non-degenerate.

Let $\xi \in \Sigma$ and let $a \in \Gamma$ be such that $a \cdot \xi = h_K^+(\xi)$. The hyperplane $x \cdot \xi = h_K^+(a)$ is tangent to Γ and hence the external normal unit vector $\gamma(a) = \nu_\Gamma(a) = \xi$. Since ξ is a regular value of γ, the point a is non-degenerate, i.e., $\kappa_\Gamma(a) \neq 0$. Applying a suitable translation and an orthogonal transformation, we can make $a = 0$ and $\xi = (0, \dots, 0, 1)$. Then the tangent plane $T_a(\Gamma)$ is the coordinate plane $x_n = 0$ and the domain K is contained in the half-space $x_n \leq 0$. In this case $h_K^+(\xi) = 0$. Moreover, after performing a suitable non-degenerate linear transformation we can make the equation of Γ, near $a = 0$, to be:

$$x_n = -\frac{1}{2}\left(c_1 x_1^2 + \dots + c_{n-1}x_{n-1}^2\right) + o\left(|x'|^2\right),\ (x_1, \dots, x_{n-1}) = x' \to 0. \tag{7}$$

The new axes x_j, $j = 1, \dots, n-1$, are the directions of the vectors of principal curvatures and the coefficients c_j are the values of the principal curvatures at the point $a = 0 \in \Gamma$. The Gaussian curvature at $a = 0$ is $\kappa_\Gamma(0) = c_1 \cdots c_{n-1}$. All the applied transformations preserve regular points, hence $\kappa_\Gamma(0) \neq 0$. Therefore, none of c_j's are equal to zero, and, since $c_j \geq 0$ due to the convexity of Γ, we have $c_j > 0$ for all j.

After the above transformations we have $\xi = (0, \dots, 0, 1)$, so the hyperplane $x \cdot \xi = t$ is now given by the equation $x_n = t$, with $t < 0$. The main term of $\mathrm{Vol}_{n-1}(K \cap \{x_n = t\})$ near $t = 0$ is determined by the main term of the expansion (7), i.e., by the volume of the ellipsoid $-2t = c_1 x_1^2 + \dots + c_{n-1}x_{n-1}^2$, which is equal to $c(-t)^{\frac{n-1}{2}}$, where $c = \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})\sqrt{\kappa_\Gamma(a)}}$.

Thus, for the specific choice $a = 0$ and $\xi = (0, \dots, 0, 1)$, we have the following asymptotic formula:

$$A_K(\xi,t) = \mathrm{Vol}_{n-1}(K \cap \{x_n = t\}) = c\,(-t)^{\frac{n-1}{2}} + o(|t|^{\frac{n-1}{2}}), t \to -0,$$

near (ξ, t_0) with $\xi = (0, 0, ..., 0, 1)$ and $t_0 = h_K^+(\xi) = 0$. Performing the inverse affine transformation, we obtain the first asymptotic formula in Lemma 2.1, with some new nonzero constant c^+ depending, of course, on ξ.

The second asymptotic relation follows from the first one and from the relations $h_K^+(-\xi) = -h_K^-(\xi)$, $A_K(-\xi,-t) = A_K(\xi,t)$. $\square$

Lemma 2.1 implies an explicit form of the quadratic polynomial q in conditions (i), (ii) of Theorem 1.4, as follows:

Lemma 2.2. *Let $n \geq 2$ be an even integer, and let K be a bounded convex body in $\mathbb{R}^n$ with C^∞ boundary ∂K. Let $q(\xi,t)$ be a quadratic polynomial of t in condition (i) or in condition (ii) of Theorem 1.4. Then $q(\xi,t) = q_0(\xi)\big(h_K^+(\xi) - t\big)\big(t - h_K^-(\xi)\big)$.*

Proof. Let $n = 2m$. Let us start with the case (ii):

$$\sqrt{q(\xi,t)}A_K(\xi,t) = P(\xi,t), \tag{8}$$

where P is a polynomial in t.

By Lemma 2.1, there is a dense set $\Sigma \in S^{n-1}$ such that the function $t \to A_K^2(\xi,t)$ vanishes at the points $h_K^{\pm}(\xi)$ (when $\xi \in \pm\Sigma$, respectively) to the order exactly $2\frac{n-1}{2} = 2m-1$. Therefore, for any $\xi \in \Sigma$ we have

$$P^2(\xi,t) = q(\xi,t)A^2(\xi,t) = q(\xi,t)\big(h_K^+(\xi) - t\big)^{2m-1}P_0(\xi,t),$$

where $P_0(\xi,t)$ is another polynomial with respect to t and $P_0\big(\xi, h_K^+(\xi)\big) \neq 0$. Then $P^2(\xi,t)$ has zero at $t = h_K^+(\xi)$, of even multiplicity. Comparing the multiplicities at both sides of the equality, we obtain $q(\xi, h_K^+(\xi)) = 0$. Since Σ is a dense subset of S^{n-1} and $q(\xi,t), h_K^+(\xi)$ are continuous with respect to ξ, this is true for all $\xi \in S^{n-1}$.

A similar argument using the expansion from Lemma 2.1 at the point $h_K^-(\xi)$ implies that $q(\xi, h_K^-(\xi)) = 0, \xi \in S^{n-1}$. Since $q(\xi,t)$ is a quadratic polynomial in t, the needed presentation for $q(\xi,t)$ follows.

The case (i) easily reduces to (ii). Indeed, if $A_K(\xi,t) = \sqrt{q(\xi,t)}P(\xi,t)$ then $\sqrt{q(\xi,t)}A_K(\xi,t) = q(\xi,t)P(\xi,t)$ and this is the case (ii) because in the right hand side we have a polynomial in t. The lemma is proved. $\square$

2.2. Functions with polynomial Hilbert transform on a finite interval

We will need some facts about the Hilbert transform (3). This transform is originally defined on continuous functions with sufficiently fast decay at infinity, but can be extended to less decaying functions and also to distributions. The Hilbert transform $\mathcal{H}$ is self-invertible; more precisely $\mathcal{H}(\mathcal{H}F) = -F$. We have the following intertwining relation between the transform $\mathcal{H}$ and the operator of multiplication by the independent variable (see [5, Section 4.7]):

$$\mathcal{H}(s\varphi(s))(t) = t\mathcal{H}\varphi(t) - \frac{1}{\pi}\int\limits_{\mathbb{R}} \varphi(s)\,ds. \tag{9}$$

Let $\chi_{[a,b]}(s)$ be the characteristic function of the interval $[a,b]$. The Hilbert transform of the function $\chi_{[-1,1]}(s)\sqrt{(1-s)(1+s)}$ is well-known (see [5, formula 11.343]):

$$\mathcal{H}\left(\chi_{[-1,1]}(s)\sqrt{(1-s)(1+s)}\right)(t) = t, \quad t \in [-1,1].$$

By a linear change of variables one obtains the Hilbert transform of $\chi_{[a,b]}(s)\sqrt{(b-s)(s-a)}$:

$$\mathcal{H}\left(\chi_{[a,b]}(s)\sqrt{(b-s)(s-a)}\right)(t) = t - \frac{b+a}{2}, \quad t \in [a,b]. \tag{10}$$

We will also make use of the inversion formula for the Hilbert transform on a finite interval (**finite Hilbert transform**). Namely, if a continuous function $F(t)$ is supported on an interval $[a,b]$, then F can be recovered from the knowledge of the values of its Hilbert transform only on $[a,b]$. The corresponding inversion formula looks as follows (see, e.g., [9]):

$$\sqrt{(b-t)(t-a)}F(t) = -\mathcal{H}\Big(\chi_{[a,b]}(s)\mathcal{H}F(s)\sqrt{(b-s)(s-a)}\Big)(t) + \frac{1}{\pi}\int_a^b F(s)\, ds,\ t \in [a,b]. \tag{11}$$

Lemma 2.3. *Let $[a,b]$ be a segment on the real line and let F be a continuous function on the real line, supported in the segment $[a,b]$. Then the following properties are equivalent:*

(a) The function $\sqrt{(b-t)(t-a)}F(t)$ is a polynomial on the interval $t \in (a,b)$.
(b) The function $\frac{F(t)}{\sqrt{(b-t)(t-a)}}$ is a polynomial on the interval $t \in (a,b)$.
(c) The Hilbert transform $\mathcal{H}F(t)$ is a polynomial on the interval $t \in (a,b)$.

Proof. $(a) \Leftrightarrow (b)$

If (a) holds then $\sqrt{(b-t)(t-a)}F(t) = Q(t),\ t \in (a,b)$, where Q is a polynomial. Since $F(t)$ is continuous at $t = a$ and $t = b$ we have $Q(a) = Q(b) = 0$, and by Bezout's theorem $Q(t) = (b-t)(t-a)Q_1(t)$, where Q_1 is another polynomial. Thus, $\sqrt{(b-t)(t-a)}F(t) = (b-t)(t-a)Q_1(t)$, $t \in (a,b)$ and hence $F(t) = \sqrt{(b-t)(t-a)}Q_1(t)$, $t \in (a,b)$, which is exactly condition (b).

Conversely, if (b) holds then $F(t) = \sqrt{(b-t)(t-a)}Q(t)$, $t \in (a,b)$, Q is a polynomial. Multiplying both sides by $\sqrt{(b-t)(t-a)}$ leads to $\sqrt{(b-t)(t-a)}F(t) = (b-t)(a-t)Q(t)$, $t \in (a,b)$, and therefore (a) holds.

$(c) \Rightarrow (a)$

Suppose that $\mathcal{H}F(t) = P(t),\ t \in (a,b)$, where P is a polynomial.

Then inversion formula (11) reads as

$$\sqrt{(b-t)(t-a)}F(t) = -\mathcal{H}\Big(\chi_{[a,b]}(s)P(s)\sqrt{(b-s)(s-a)}\Big)(t) + \frac{1}{\pi}\int_a^b F(s)\, ds,\ t \in [a,b], \tag{12}$$

and therefore, to prove (a), it suffices to prove that the right hand side is a polynomial on the interval (a,b). In turn, it suffices to check this only for monomials $P(s) = s^k$.

Thus, we need to prove that the Hilbert transform of the function $\chi_{[a,b]}(s)s^k\sqrt{(b-s)(s-a)}$ is a polynomial.

It is true for $k = 0$ because identity (10) yields

$$\mathcal{H}\Big(\chi_{[a,b]}(s)\sqrt{(b-s)(s-a)}\Big)(t) = t + c_0,\ t \in [a,b],$$

for a certain constant c_0. For $k > 0$ formula (11) leads to

$$\mathcal{H}\Big(\chi_{[a,b]}(s)s^k\sqrt{(b-s)(s-a)}\Big)(t) = t\Big[\mathcal{H}\Big(\chi_{[a,b]}(s)s^{k-1}\sqrt{(b-s)(s-a)}\Big)(t) + c_k\Big],$$

where c_k is a constant. Thus, by induction, the above two equalities imply that

$$\mathcal{H}\Big(\chi_{[a,b]}(s)s^k\sqrt{(b-s)(s-a)}\Big)(t)$$

is a polynomial of degree $k+1$. Thus, the right hand side in (12) is a polynomial when $P(s)$ is a monomial of an arbitrary degree and hence this is true for any polynomial P which proves (a).

$(b) \Rightarrow (c)$

If (b) is fulfilled then $F(t) = \sqrt{(b-t)(t-a)}Q(t),\ t \in (a,b)$ for some polynomial $Q(t)$. Then inversion formula (11) for the finite Hilbert transform on $[a,b]$ can be written as

$$(b-t)(t-a)Q(t) = -\mathcal{H}\left(\chi_{[a,b]}(s)\mathcal{H}F(s)\sqrt{(b-s)(s-a)}\right)(t) + c_1,\ t \in [a,b],$$

where c_1 is a constant.

Denote for convenience $G(s) = \chi_{[a,b]}(s)\mathcal{H}F(s)\sqrt{(b-s)(s-a)}$. Then for $t \in [a,b]$ we have

$$\mathcal{H}G(t) = Q_1(t),$$

where $Q_1(t) = -(b-t)(t-a)Q(t) + c_1$. Again, inversion formula (11) yields:

$$G(t)\sqrt{(b-t)(t-a)} = -\mathcal{H}\left(\chi_{[a,b]}(s)\sqrt{(b-s)(s-a)}Q_1(s)\right)(t) + c_2,$$

with some constant c_2. We have just proven that the expression in the right hand side is a polynomial on $t \in [a,b]$.

Substituting the expression for G we arrive at

$$(b-t)(t-a)\mathcal{H}F(t) = P(t),\ t \in (a,b),$$

where $P(t)$ is a polynomial. Since $F(t)$ is bounded on the real line, $|F(t)| \le C$, and supported in $[a,b]$, its Hilbert transform satisfies $|\mathcal{H}F(t)| \le \frac{C}{\pi}\ \ln\frac{b-t}{t-a},\ t \in (a,b)$. Hence the limits, as $t \to a,\ t \to b$, of the left hand side of the above equality are equal to zero. This implies $P(a) = P(b) = 0$ and hence $P(t) = (b-t)(a-t)P_1(t)$, where P_1 is a polynomial. Then $\mathcal{H}F(t) = P_1(t),\ t \in (a,b)$ and property (c) is proved. Thus, we have proven that the properties (a), (b), (c) are equivalent. The Lemma is proved. □

2.3. Equivalence of conditions $(i), (ii), (iii)$

The equivalence of conditions $(i), (ii)$ and (iii) of Theorem 1.4 follows immediately from Lemma 2.2 and also from Lemma 2.3 applied to $F(t) = A_K(\xi,t),\ a = h_K^-(\xi),\ b = h_K^+(\xi)$. Indeed, Lemma 2.2 gives an explicit form of the quadratic polynomial $q(\xi,t)$ in $(i), (ii)$ and says that conditions $(i), (ii), (iii)$ for $A_K(\xi,t)$ read as conditions $(a), (b), (c)$, respectively, for the function $F(t)$ in Lemma 2.3. The latter lemma claims that conditions $(a), (b), (c)$ are equivalent and therefore conditions $(i), (ii), (iii)$ are equivalent, too.

3. Proof of Theorem 1.4. Equivalence of conditions (iii) and (iv)

Let us first show that (iv) implies (iii). Suppose that (iv) holds, i.e., K is an ellipsoid. Applying a translation, if needed, we may assume that the center of the ellipsoid is at the origin, and therefore its section function $A_K(\xi,t)$ is given by (4). Since n is even, $A_K(\xi,t)$ satisfies (ii) with $q(\xi,t) = h_K^2(\xi) - t^2$. It suffices to notice that, as we have proven in the previous section, conditions (ii) and (iii) are equivalent.

We will now prove that (iii) implies (iv). Before we start, let us outline the plan of the proof. Let K be a convex body satisfying (iii). Without loss of generality we may assume that the origin is an interior point of K. Since $\mathcal{H}A_K(\xi,t)$ is a polynomial in t of degree at most N, the derivatives of $\mathcal{H}A_K(\xi,t)$ with respect to t of orders greater than N at $t = 0$ are equal to zero. In order to find derivatives of $\mathcal{H}A_K(\xi,t)$ at

$t = 0$, we will compute its fractional derivatives. The reader is referred to [6, Section 2.6] for more details about such techniques. The next step is to express fractional derivatives of $\mathcal{H}A_K(\xi, t)$ at zero in terms of the Fourier transform of expressions involving powers of the Minkowski functional of K. Recall that the latter is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \qquad x \in \mathbb{R}^n.$$

Since ordinary derivatives are obtained by computing fractional derivatives at positive integers, we will get the condition that the Fourier transform of $\|-x\|_K^{-n+1+m} + (-1)^{m+1}\|x\|_K^{-n+1+m}$ must be concentrated at the origin for large enough integers m. This implies that $\|-x\|_K^{-n+1+m} + (-1)^{m+1}\|x\|_K^{-n+1+m}$ must be a homogeneous polynomial of x. An algebraic result from [7] then implies that K must be an ellipsoid in even dimensions.

Now we will provide details of the above plan. Let us write the Hilbert transform of $A_K(\xi, t)$ as follows

$$\mathcal{H}A_K(\xi, t) = \frac{1}{\pi} \int_0^\infty \frac{A_K(\xi, t-z) - A_K(\xi, t+z)}{z}\, dz.$$

Let q be a complex number such that $-1 < \Re q < 0$. Consider the fractional derivative of order q at $t = 0$ of the function $\mathcal{H}A_K(\xi, t)$.

$$(\mathcal{H}A_K)^{(q)}(\xi, 0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q}\, \mathcal{H}A_K(\xi, -t)\, dt.$$

Let us briefly explain why the last integral converges. $\mathcal{H}A_K(\xi, t)$ is a continuous function of t on $\mathbb{R}$ except possibly at the points $t = h_K^+(\xi)$ and $t = h_K^-(\xi)$, where in the worst case it behaves as $\ln|h_K^+(\xi) - t|$ and $\ln|h_K^-(\xi) - t|$ respectively. Additionally, as $t \to \pm\infty$ it behaves as $1/t$.

Writing $\mathcal{H}A_K$ as follows:

$$\mathcal{H}A_K(\xi, t) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_0^\infty \frac{A_K(\xi, t-z) - A_K(\xi, t+z)}{z^{1+\epsilon}}\, dz,$$

and using the dominated convergence theorem and Fubini's theorem we get

$$\begin{aligned}
(\mathcal{H}A_K)^{(q)}(\xi, 0) &= \lim_{\epsilon \to 0^+} \frac{1}{\pi\Gamma(-q)} \int_0^\infty t^{-1-q} \int_0^\infty \frac{A_K(\xi, -t-z) - A_K(\xi, -t+z)}{z^{1+\epsilon}}\, dz\, dt \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\pi\Gamma(-q)} \int_0^\infty \frac{1}{z^{1+\epsilon}} \int_0^\infty t^{-1-q} \left(A_K(\xi, -t-z) - A_K(\xi, -t+z)\right) dt\, dz \\
&= \lim_{\epsilon \to 0^+} \frac{1}{\pi\Gamma(-q)} \int_0^\infty \frac{1}{z^{1+\epsilon}} \int_{\mathbb{R}^n} \Big((-z - x\cdot\xi)_+^{-1-q} - (z - x\cdot\xi)_+^{-1-q} \Big) \chi(\|x\|_K)\, dx\, dz.
\end{aligned}$$

Here and below we use the following notation. If $\Re\lambda > -1$, then

$$t_+^\lambda = \begin{cases} 0, & t \leq 0, \\ t^\lambda, & t > 0, \end{cases} \qquad \text{and} \qquad t_-^\lambda = \begin{cases} |t|^\lambda, & t < 0, \\ 0, & t \geq 0. \end{cases}$$

Observe that $(\mathcal{H}A_K)^{(q)}(\xi,0)$ naturally extends to a homogeneous function of $\xi\in\mathbb{R}^n$ of degree $-1-q$, and we will consider its distributional Fourier transform with respect to ξ. Let ϕ be a Schwarz function. Then

$$\begin{aligned}\langle\Big((\mathcal{H}A_K)^{(q)}(\cdot,0)\Big)^{\wedge},\phi\rangle &= \langle(\mathcal{H}A_K)^{(q)}(\xi,0),\hat\phi(\xi)\rangle\\ &= \lim_{\epsilon\to0^+}\frac{1}{\pi\Gamma(-q)}\int_0^\infty\frac{1}{z^{1+\epsilon}}\int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{\mathbb{R}^n}\Big((-z-x\cdot\xi)_+^{-1-q}-(z-x\cdot\xi)_+^{-1-q}\Big)\hat\phi(\xi)\,d\xi\,dxdz\\ &= \lim_{\epsilon\to0^+}\frac{1}{\pi\Gamma(-q)}\int_0^\infty\frac{1}{z^{1+\epsilon}}\int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{\mathbb{R}}\Big((u-z)_+^{-1-q}-(u+z)_+^{-1-q}\Big)\int_{x\cdot\xi=-u}\hat\phi(\xi)\,d\xi\,du\,dxdz.\end{aligned}$$

The Fourier transform of $(u-z)_+^{-1-q}-(u+z)_+^{-1-q}$ with respect to u equals

$$\begin{aligned}&i\Gamma(-q)\Big(e^{i(-1-q)\pi/2}s_+^q e^{-izs}-e^{i(1+q)\pi/2}s_-^q e^{-izs}-e^{i(-1-q)\pi/2}s_+^q e^{izs}+e^{i(1+q)\pi/2}s_-^q e^{izs}\Big)\\ &\qquad\qquad= 2\sin(zs)\Gamma(-q)\Big(e^{i(-1-q)\pi/2}s_+^q-e^{i(1+q)\pi/2}s_-^q\Big);\end{aligned}$$

see [4, Ch. II, Sec. 2.3].

Using the connection between the Radon transform and the Fourier transform, we get

$$\begin{aligned}&(2\pi)^{-n+1}\langle\Big((\mathcal{H}A_K)^{(q)}(\xi,0)\Big)^{\wedge},\phi\rangle\\ &\quad= \lim_{\epsilon\to0^+}\frac{2}{\pi}\int_0^\infty\frac{1}{z^{1+\epsilon}}\int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{\mathbb{R}}\sin(zs)\Big(e^{i(-1-q)\pi/2}s_+^q-e^{i(1+q)\pi/2}s_-^q\Big)\phi(-sx)\,ds\,dx\,dz\\ &\quad= \lim_{\epsilon\to0^+}\frac{2}{\pi}\int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{\mathbb{R}}\int_0^\infty\frac{\sin(zs)}{z^{1+\epsilon}}\,dz\Big(e^{i(-1-q)\pi/2}s_+^q-e^{i(1+q)\pi/2}s_-^q\Big)\phi(-sx)\,ds\,dx.\end{aligned}$$

The latter use of the Fubini theorem explains why we passed from $1/z$ to $1/z^{1+\epsilon}$ earlier: the integral of $\frac{\sin(zs)}{z^{1+\epsilon}}$ is absolutely convergent, while the integral of $\frac{\sin(zs)}{z}$ is not. To compute $\int_0^\infty\frac{\sin(zs)}{z^{1+\epsilon}}\,dz$ we can write it as $\frac{1}{2i}\int_0^\infty\frac{e^{izs}-e^{-izs}}{z^{1+\epsilon}}\,dz$ and then repeat the calculations from [4, Ch. II, Sec. 2.3] for the Fourier transform of $z_+^{-1-\epsilon}$. As a result we get

$$\begin{aligned}\int_0^\infty\frac{\sin(zs)}{z^{1+\epsilon}}\,dz &= \frac{1}{2i}\left(ie^{-i(1+\epsilon)\pi/2}\Gamma(-\epsilon)\left(s_+^\epsilon+e^{i\epsilon\pi}s_-^\epsilon-s_-^\epsilon-e^{i\epsilon\pi}s_+^\epsilon\right)\right)\\ &= \frac{1}{2}e^{-i(1+\epsilon)\pi/2}\Gamma(-\epsilon)\left(s_+^\epsilon-s_-^\epsilon\right)\left(1-e^{i\epsilon\pi}\right)\\ &= -\sin(\epsilon\pi/2)\Gamma(-\epsilon)|s|^\epsilon\operatorname{sgn}(s).\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to0^+}\int_0^\infty\frac{\sin(zs)}{z^{1+\epsilon}}\,dz=\frac{\pi}{2}\operatorname{sgn}(s),$$

and hence

$$\begin{aligned}
&(2\pi)^{-n+1}\langle\Big(\big((\mathcal{H}A_K)^{(q)}(\xi,0)\big)^\wedge, \phi\rangle \\
&= \int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{\mathbb{R}}\operatorname{sgn}(s)\Big(e^{i(-1-q)\pi/2}s_+^q - e^{i(1+q)\pi/2}s_-^q\Big)\phi(-sx)\,ds\,dx \\
&= \int_{\mathbb{R}^n}\chi(\|x\|_K)\int_0^\infty \operatorname{sgn}(s)e^{i(-1-q)\pi/2}s^q\,ds\,\phi(-sx)\,dx \\
&\quad - \int_{\mathbb{R}^n}\chi(\|x\|_K)\int_{-\infty}^0 \operatorname{sgn}(s)e^{i(1+q)\pi/2}|s|^q\,ds\,\phi(-sx)\,dx \\
&= \int_{\mathbb{R}^n}\chi(\|-x\|_K)\int_0^\infty e^{i(-1-q)\pi/2}s^q\,ds\,\phi(sx)\,dx \\
&\quad + \int_{\mathbb{R}^n}\chi(\|x\|_K)\int_0^\infty e^{i(1+q)\pi/2}s^q\,ds\,\phi(sx)\,dx \\
&= e^{-i(1+q)\pi/2}\int_{S^{n-1}}\int_0^{\|-\theta\|_K} r^{n-1}\int_0^\infty s^q\phi(sr\theta)\,ds\,dr\,d\theta \\
&\quad + e^{i(1+q)\pi/2}\int_{S^{n-1}}\int_0^{\|\theta\|_K} r^{n-1}\int_0^\infty s^q\phi(sr\theta)\,ds\,dr\,d\theta \\
&= e^{-i(1+q)\pi/2}\int_{S^{n-1}}\int_0^{\|-\theta\|_K} r^{n-2-q}\int_0^\infty s^q\phi(s\theta)\,ds\,dr\,d\theta \\
&\quad + e^{i(1+q)\pi/2}\int_{S^{n-1}}\int_0^{\|\theta\|_K} r^{n-2-q}\int_0^\infty s^q\phi(s\theta)\,ds\,dr\,d\theta \\
&= \frac{1}{n-1-q}\int_{S^{n-1}}\Big(e^{-i(1+q)\pi/2}\|-\theta\|_K^{-n+1+q} + e^{i(1+q)\pi/2}\|\theta\|_K^{-n+1+q}\Big)\int_0^\infty s^q\phi(s\theta)\,ds\,d\theta \\
&= \frac{1}{n-1-q}\int_{\mathbb{R}^n}\Big(e^{-i(1+q)\pi/2}\|-x\|_K^{-n+1+q} + e^{i(1+q)\pi/2}\|x\|_K^{-n+1+q}\Big)\phi(x)\,dx.
\end{aligned}$$

Thus, we have shown that

$$(2\pi)^{-n+1}\Big((\mathcal{H}A_K)^{(q)}(\xi,0)\Big)^\wedge(x) = \frac{1}{n-1-q}\Big(e^{-i(1+q)\pi/2}\|-x\|_K^{-n+1+q} + e^{i(1+q)\pi/2}\|x\|_K^{-n+1+q}\Big),$$

that is

$$(\mathcal{H}A_K)^{(q)}(\xi,0) = \frac{1}{2\pi(n-1-q)}\Big(e^{-i(1+q)\pi/2}\|-x\|_K^{-n+1+q} + e^{i(1+q)\pi/2}\|x\|_K^{-n+1+q}\Big)^\wedge(\xi),$$

for all complex q such that $-1 < \Re q < 0$. Using analytic continuation, we see that the formula is still valid for all $q \in \mathbb{C}\setminus\{n-1\}$.

Since $\mathcal{H}A_K(\xi,t)$ is a polynomial of t of degree at most N, we have $(\mathcal{H}A_K)^{(m)}(\xi,0)=0$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$ and all natural $m>\max\{N,n-1\}$. This means that

$$\left(e^{-i(1+m)\pi/2}\|-x\|_K^{-n+1+m}+e^{i(1+m)\pi/2}\|x\|_K^{-n+1+m}\right)^{\wedge}(\xi)$$

is a linear combination of derivatives of the delta function supported at the origin. Thus,

$$e^{-i(1+m)\pi/2}\|-x\|_K^{-n+1+m}+e^{i(1+m)\pi/2}\|x\|_K^{-n+1+m}$$

is a polynomial of x.

When m is odd, we get

$$e^{-i(1+m)\pi/2}=e^{i(1+m)\pi/2},$$

and hence

$$\|-x\|_K^{-n+1+m}+\|x\|_K^{-n+1+m}$$

is a polynomial when m is odd.

Similarly, when m is even, we get

$$e^{-i(1+m)\pi/2}=-e^{i(1+m)\pi/2},$$

and thus

$$\|-x\|_K^{-n+1+m}-\|x\|_K^{-n+1+m}$$

is a polynomial when m is even.

Thus for every positive integer $\ell>(N-n)/2$ we have

$$\|-x\|_K^{2\ell+1}-\|x\|_K^{2\ell+1}=P_\ell(x)$$

and

$$\|-x\|_K^{4\ell+2}+\|x\|_K^{4\ell+2}=Q_\ell(x),$$

for some homogeneous polynomials P_ℓ and Q_ℓ of degrees $2\ell+1$ and $4\ell+2$ respectively. Solving the latter system of equations we obtain

$$\|x\|_K^{2\ell+1}=\widetilde{P}_\ell(x)+\sqrt{\widetilde{Q}_\ell(x)}, \tag{13}$$

where $\widetilde{P}_\ell(x)=-\frac{1}{2}P_\ell(x)$ and $\widetilde{Q}_\ell(x)=\frac{1}{4}\left(2Q_\ell(x)-P_\ell^2(x)\right)$.

As was shown in [7, Theorem 3.6], condition (13) for all large ℓ implies that

$$\|x\|_K=P(x)+\sqrt{Q(x)}, \tag{14}$$

for a linear polynomial P and a positive quadratic polynomial Q. From here it is not difficult to see that K must be an ellipsoid. Indeed, let $x\in\partial K$, then $\|x\|_K=1$ and therefore (14) yields

$$(1-P(x))^2=Q(x).$$

This means that ∂K is a quadric hypersurface. Since K is compact, it must be an ellipsoid.

Finally, let us remark that bodies with polynomial $\mathcal{H}A_K(\xi, t)$ do not exist in odd dimensions. This follows from the fact that the function

$$\| - x\|_K^{-n+1+m} + \|x\|_K^{-n+1+m}$$

is an even function, but at the same time, it has to be a polynomial of an odd degree $-n+m+1$, if n and m are both odd. The only polynomial that is both odd and even is the zero polynomial.

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