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A solution to the fifth and the eighth Busemann-Petty problems in a small neighborhood of the Euclidean ball ☆

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ABSTRACT

We show that the fifth and the eighth Busemann-Petty problems have positive solutions for bodies that are sufficiently close to the Euclidean ball in the Banach-Mazur distance.

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1. Introduction

In [3] Busemann and Petty posed 10 problems concerning the plane sections through the center of a symmetric convex body in $\mathbb{R}^n$, which arose in their study of Minkowski spaces. They expressed the belief that "contributions to these problems would not only advance the theory of Minkowski spaces,[1] but lead the way to a new direction of research on convex bodies". The first problem had attracted a lot of attention before its final solution appeared at the end of 1990's in a series of papers by many mathematicians (see [7]). The solution brought to existence several new methods that are widely used now, including Lutwak's dual Brunn-Minkowski theory and the Fourier analytic approach to convex geometry. We are, however, not aware of any significant work on any of the other nine problems since the time of their formulation by Busemann and Petty.

The current paper offers a modest progress in the fifth and the eighth problems. Namely, we show that the corresponding conjectures hold in a sufficiently small neighborhood of the Euclidean ball in the Banach-Mazur distance. Our approach continues the tradition of using harmonic analysis methods in convex geometry but, unfortunately, as of now, our tools work only at the level of perturbation theory. It would be extremely interesting to extend them to the general setting or to find an alternative approach.

The fifth and the eighth problems can be equivalently restated in the convex geometry language as follows.

Problem 5. If for an origin-symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, we have

$$h_K(\theta)\mathrm{vol}_{n-1}(K \cap \theta^\perp) = C \qquad \forall \theta \in S^{n-1}, \tag{1}$$

where the constant C is independent of θ, must K be an ellipsoid?

Here S^{n-1} is the unit sphere in $\mathbb{R}^n$, $\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = 0\}$ is the hyperplane orthogonal to the direction $\theta \in S^{n-1}$ and passing through the origin, and $h_K(\theta) = \max_{x \in K} \langle x, \theta \rangle$ is the support function of the convex body $K \subset \mathbb{R}^n$.

Problem 8. If for an origin-symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, we have

$$f_K(\theta) = C(\mathrm{vol}_{n-1}(K \cap \theta^\perp))^{n+1} \qquad \forall \theta \in S^{n-1}, \tag{2}$$

where the constant C is independent of θ, must K be an ellipsoid?

Here f_K is the curvature function of K. In the C^2-smooth non-degenerate case it is just the reciprocal of the Gaussian curvature viewed as a function of the unit normal vector.

[1] We refer the reader interested in the Minkowski geometry meaning of the Busemann-Petty problems to [2,3,11] and the survey [9].

The general definition is as follows. For a convex body K, let $\Omega \subset \partial K$ be the set of points x on the boundary ∂K of K where the supporting hyperplane of K and, thereby, the outer unit normal vector θ_x is unique. By Theorem 2.2.5 (see [12, pg. 84]), $\mathcal{H}^{n-1}(\partial K \setminus \Omega) = 0$. The mapping $\mathcal{G}(x) = \theta_x$ is a well-defined Borel measurable mapping from Ω to S^{n-1}. Let $\mu = \mathcal{G}_* \mathcal{H}^{n-1}|_\Omega$ be the push-forward of the surface area on ∂K to the unit sphere. When μ is absolutely continuous with respect to the surface measure on the sphere, we call its Radon-Nikodym density the curvature function and denote it by f_K. It is not hard to see that this definition coincides with the previous one in the smooth case (see [12, pg. 545]).

It is unclear to us what degree of smoothness Busemann and Petty assumed when posing Problem 8, so we will handle the general case. The reader will lose almost nothing, however, by assuming that ∂K is smooth and non-degenerate, but K is close to the unit ball only in the Banach-Mazur distance and not in C^2.

The Euclidean ball clearly satisfies (1) and (2). On the other hand, if a symmetric convex body K satisfies (1) or (2), then so does TK where T is an invertible linear map from $\mathbb{R}^n$ to itself. Hence, (1) and (2) are satisfied by ellipsoids.

Note that this linear invariance motivated the very formulations of the Busemann-Petty problems so that their statements reflect the intrinsic properties of the corresponding Minkowski spaces rather than just those of the particular embeddings of these spaces into $\mathbb{R}^n$. In particular, the power $n+1$ on the right-hand side in (2) is the only power that makes the statement of Problem 8 affine invariant.

In this paper we prove the following result.

Theorem. *Let $n \geq 3$. If an origin-symmetric convex body $K \subset \mathbb{R}^n$ satisfies (1) or (2) and is sufficiently close to the Euclidean ball in the Banach-Mazur metric, then K must be an ellipsoid.*

In dimension 2, there are convex bodies satisfying (1) that are not ellipses but, nevertheless, can be arbitrarily close to the unit disc. The curve bounding such a body is a so-called *Radon curve*, see [10], in particular, Section 6. On the other hand, the only convex bodies satisfying (2) in dimension 2 are the ellipses [11, Theorem 5.6].

2. Outline of the proof

The arguments presented in this paper are mainly inspired by and closely follow the ones in [4], where the limiting behavior of the iterations of the intersection body operator was studied in a small neighborhood of the unit ball. For the purposes of this exposition, the main result of [4] can be considered as the statement that a convex body K sufficiently close to an ellipsoid in the Banach-Mazur metric whose second intersection body is homothetic to K must be an ellipsoid.

Our first (well-known) remark is that both the fifth and the eighth Busemann-Petty problems are invariant under linear transformations (see Section 3). This allows us to consider convex bodies close to the unit ball in the Hausdorff distance rather than just in the Banach-Mazur one (see Section 4). Similarly to [4], we employ the spherical harmonic decomposition of the radial function ρ_K of the body K and choose the affine image of K for which the contribution of the second order harmonics is negligible. It turns out that the standard isotropic position (see Section 5) is already good enough from this point of view.

However in [4] these preliminary steps immediately resulted in the equation $\rho_K = I[I[\rho_K]]$ with $I[\rho] = \mathcal{R}\left[\rho^{n-1}\right]$, where $\mathcal{R}$ is the usual Radon transform on the unit sphere S^{n-1}. Then the linearization

$$\mathcal{R}\left[\rho^{n-1}\right] = 1 + (n-1)\mathcal{R}(\rho - 1) + \text{higher order terms}$$

of the right-hand side for $\rho \approx 1$ and good contracting properties of the operator $w \mapsto (n-1)\mathcal{R}w$ on the space of even functions on the sphere orthogonal to constants and quadratic polynomials essentially finished the story.

Our main difficulty now is that a similar analytic reformulation $h_K = \left(\mathcal{R}\left[\rho_K^{n-1}\right]\right)^{-1}$ (see Section 6) of the fifth Busemann-Petty problem, which is the simpler of the two, involves both the radial function ρ_K and the support function h_K of the convex body K. They uniquely determine each other, of course, but the relation between them is non-linear, non-smooth, and quite complicated in general, so the straight-forward elementary smooth calculus techniques seem no longer directly applicable.

What saves the day, however, is that when K is close to the unit ball, h_K can essentially differ from ρ_K only if both experience noticeable (compared to the full L^2-norm of their respective deviations from constants) high frequency oscillations. The above equation does not prohibit it for ρ_K but, since it gives an expression for h_K involving the Radon transform and the latter suppresses high frequencies, it makes it impossible for h_K. The formal expressions of these two steps are given by Lemmas 3 and 4 (see Sections 7 and 8). Their formulations are a bit too long and technical to include into this brief outline, so we refer the interested reader directly to the main text here.

Problem 8 presents an extra complication that even the left-hand side in the corresponding analytic reformulation is given by a nonlinear second order differential operator of h_K, so, in addition to the linearization techniques and the above observations, we need to use some elementary theory of the Monge-Ampére equation (see Appendix II) there.

While we treat Problems 5 and 8 in parallel whenever possible, the reader who wants to get the gist of our present work without going into the nitty-gritty of it may choose to ignore everything related to Problem 8 entirely and to concentrate exclusively on Problem 5 postponing the rest for later readings.

3. Invariance of Busemann-Petty problems under linear transformations

Both Busemann-Petty problems are invariant under linear transformations. More precisely, we have the following

Lemma 1. *If a body K satisfies (1), then so does TK for any linear transformation $T \in GL(n)$ with constant $C \cdot |\det T|$. Similarly, if a body K satisfies (2), then so does TK for any linear transformation $T \in GL(n)$ with constant $C \cdot |\det T|^{1-n}$.*

Proof. This statement is almost obvious for Problem 5. Indeed, let H be any hyperplane in $\mathbb{R}^n$ passing through the origin and let H_s be a support hyperplane of K parallel to H. Consider any point $p \in K \cap H_s$ and the cone $C_{K,H}$ with the base $K \cap H$ and the vertex p. Note that due to the symmetry of K and the fact that H_s is parallel to H, the volume $\mathrm{vol}_n(C_{K,H})$ of this cone is independent of the particular choice of H_s and p. Moreover, we clearly have $C_{TK,TH} = T(C_{K,H})$, so $\mathrm{vol}_n(C_{TK,TH}) = |\det T|\mathrm{vol}_n(C_{K,H})$. Since for $H = \theta^\perp$ this volume can be expressed as $\mathrm{vol}_n(C_{K,H}) = \frac{1}{n}\mathrm{vol}_{n-1}(K \cap \theta^\perp)h_K(\theta)$, we see that property (1) is merely the statement that $\mathrm{vol}_n(C_{K,H})$ is independent of the choice of the hyperplane H (this was exactly how the fifth Busemann-Petty problem was originally formulated in [3]).

The invariance of (2) under linear transformations is somewhat less transparent. When K has smooth non-degenerate C^2-boundary with strictly positive Gaussian curvature at each point, we can restate it as follows.

Let, as before, H be an arbitrary hyperplane passing through the origin, let H_s be one of the two supporting hyperplanes of K parallel to H, and let $p \in K \cap H_s$. For $t \in (0,1)$, let H^t be the hyperplane between H and H_s parallel to H such that the distance between H_s and H^t is t times the distance d between H_s and H. Then, for small t, the $(n-1)$-dimensional volume $\mathrm{vol}_{n-1}(K \cap H^t)$ is approximately proportional to $\frac{t^{\frac{n-1}{2}} d^{\frac{n-1}{2}}}{\sqrt{G(p)}}$ where $G(p)$ is the Gaussian curvature of ∂K at p.

Note now that $\frac{\mathrm{vol}_{n-1}(K\cap H^t)}{\mathrm{vol}_{n-1}(K\cap H)}$ is invariant under linear transformations and $d\,\mathrm{vol}_{n-1}(K\cap H) = n\,\mathrm{vol}_n(C_{K,H})$ is multiplied by $|\det T|$ when we replace K by TK and H by TH. Thus, $G(p)\mathrm{vol}_{n-1}(K\cap H)^{n+1}$ equals (up to a constant factor depending on the dimension n only)

$$\lim_{t\to 0} t^{n-1}\mathrm{vol}_n(C_{K,H})^{n-1}\left[\frac{\mathrm{vol}_{n-1}(K \cap H)}{\mathrm{vol}_{n-1}(K \cap H^t)}\right]^2,$$

and thus is multiplied by $|\det T|^{n-1}$ when we replace K by TK and H by TH.

The formal proof of the invariance of (2) under linear transformations in the general case can be found in [8], see Proposition 2.9 on page 52. □

4. From the Banach-Mazur distance to the Hausdorff one

Applying an appropriate linear transformation, we can assume that the constants in (1) and (2) are equal to those for the unit ball B_2^n and that $(1-\varepsilon)rB_2^n \subset K \subset (1+\varepsilon)rB_2^n$ for some $r > 0$ with some small $\varepsilon > 0$.

Our task here will be to show that r must be close to 1, i.e., K must be close to the unit Euclidean ball B_2^n in the Hausdorff metric. We have

$$(1-\varepsilon)rh_{B_2^n} \leq h_K \leq (1+\varepsilon)rh_{B_2^n} \tag{3}$$

and

$$(1-\varepsilon)^{n-1}r^{n-1}\text{vol}_{n-1}(B_2^n \cap \theta^\perp) \leq \text{vol}_{n-1}(K \cap \theta^\perp) \leq \\ (1+\varepsilon)^{n-1}r^{n-1}\text{vol}_{n-1}(B_2^n \cap \theta^\perp). \tag{4}$$

In the case of (1), combining (3) and (4) with the equation

$$h_K(\theta)\text{vol}_{n-1}(K \cap \theta^\perp) = h_{B_2^n}(\theta)\text{vol}_{n-1}(B_2^n \cap \theta^\perp),$$

we obtain $(1-\varepsilon)^n r^n \leq 1 \leq (1+\varepsilon)^n r^n$, i.e., $\frac{1}{1+\varepsilon} \leq r \leq \frac{1}{1-\varepsilon}$.

In the case of (2), we can integrate both sides with respect to the $(n-1)$-dimensional Lebesgue measure on S^{n-1} to conclude (see [12], Section 5.3.1) that

$$\Sigma(K) = \int\limits_{S^{n-1}} f_K(\theta)dm_{n-1}(\theta) = \\ c_n \int\limits_{S^{n-1}} \left(\text{vol}_{n-1}(K \cap \theta^\perp)\right)^{n+1} dm_{n-1}(\theta), \tag{5}$$

where $\Sigma(K)$ is the surface area of ∂K and c_n is defined by

$$\Sigma(B_2^n) = c_n \int\limits_{S^{n-1}} \left(\text{vol}_{n-1}(B_2^n \cap \theta^\perp)\right)^{n+1} dm_{n-1}(\theta).$$

From our assumption $(1-\varepsilon)rB_2^n \subset K \subset (1+\varepsilon)rB_2^n$, it follows that

$$(1-\varepsilon)^{n-1}r^{n-1}\Sigma(B_2^n) \leq \Sigma(K) \leq (1+\varepsilon)^{n-1}r^{n-1}\Sigma(B_2^n),$$

which, together with (4) and (5), gives

$$(1-\varepsilon)^{n-1}r^{n-1} \leq (1+\varepsilon)^{(n-1)(n+1)}r^{(n-1)(n+1)}$$

and

$$(1+\varepsilon)^{n-1}r^{n-1} \geq (1-\varepsilon)^{(n-1)(n+1)}r^{(n-1)(n+1)},$$

i.e.,

$$\frac{1-\varepsilon}{(1+\varepsilon)^{n+1}} \leq r^n \leq \frac{1+\varepsilon}{(1-\varepsilon)^{n+1}}.$$

5. The isotropic position

We have seen in the previous section that, without loss of generality, we may assume that $(1-\varepsilon)B_2^n \subset K \subset (1+\varepsilon)B_2^n$. However, this requirement still leaves some freedom as to what affine image of K to choose. In this section we will reduce this freedom even further by putting K into the so-called isotropic position, i.e., the position where

$$\int_K \langle x, y\rangle^2 dy = c\,|x|^2 \qquad\qquad \forall x \in \mathbb{R}^n.$$

The existence of such a position is well known and easy to derive (see [1], Section 2.3.2). Indeed, for an arbitrary symmetric convex body K, the mapping

$$\mathbb{R}^n \ni x \mapsto \int_K \langle x, y\rangle^2 dy = \sum_{i,j} \Big(\int_K y_i y_j dy \Big) x_i x_j$$

is a positive-definite quadratic form. Thus, it can be written as $\langle Sx, x\rangle$, where S is a self-adjoint positive definite operator on $\mathbb{R}^n$.

Moreover, if $K = B_2^n$, then $S = c_n I$ for some $c_n > 0$. If $(1-\varepsilon)B_2^n \subset K$, then

$$\langle Sx, x\rangle = \int_K \langle x, y\rangle^2 dy \geq \int_{(1-\varepsilon)B_2^n} \langle x, y\rangle^2 dy = (1-\varepsilon)^{n+2} c_n |x|^2$$

and, similarly, if $K \subset (1+\varepsilon)B_2^n$, then

$$\langle Sx, x\rangle \leq (1+\varepsilon)^{n+2} c_n |x|^2.$$

Thus, setting $\widetilde{S} = c_n^{-1} S$, we have

$$(1-\varepsilon)^{n+2}|x|^2 \leq \langle \widetilde{S}x, x\rangle \leq (1+\varepsilon)^{n+2}|x|^2.$$

It follows that

$$(1-\varepsilon)^{n(n+2)} \leq \det(\widetilde{S}) \leq (1+\varepsilon)^{n(n+2)}$$

and

$$\|\widetilde{S}\| \leq (1+\varepsilon)^{n+2}, \quad \|\widetilde{S}^{-1}\| \leq (1-\varepsilon)^{-(n+2)},$$

whence the operator $T = \sqrt{\det(\widetilde{S})^{-\frac{1}{n}} \widetilde{S}}$ satisfies

$$\det T = 1, \qquad\qquad \|T\|, \|T^{-1}\| \leq \Big(\frac{1+\varepsilon}{1-\varepsilon} \Big)^{\frac{n+2}{2}},$$

and $T^{-1}ST^{-1}$ is a multiple of the identity.

The body $\widetilde{K} = T^{-1}K$ satisfies

$$\int_{\widetilde{K}} \langle x, y\rangle^2 dy = \int_K \langle x, T^{-1}y\rangle^2 dy = \int_K \langle T^{-1}x, y\rangle^2 dy =$$
$$\langle ST^{-1}x, T^{-1}x\rangle = \langle T^{-1}ST^{-1}x, x\rangle = c|x|^2$$

for some $c > 0$, while we also have

$$(1-\varepsilon)\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{\frac{n+2}{2}} B_2^n \subset T^{-1}(1-\varepsilon)B_2^n \subset T^{-1}K$$
$$\subset T^{-1}(1+\varepsilon)B_2^n \subset (1+\varepsilon)\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)^{\frac{n+2}{2}} B_2^n.$$

6. The analytic reformulation

Let ρ_K, $h_K : S^{n-1} \to \mathbb{R}$ be the radial and the support functions of the convex body K respectively, i.e.,

$$\rho_K(\theta) = \max\{t > 0 : \, t\theta \in K\}$$

and

$$h_K(\theta) = \max\{\langle x, \theta\rangle : \, x \in K\}.$$

The $(n-1)$-dimensional volume of the section $K \cap \theta^\perp$ is given by

$$\mathrm{vol}_{n-1}(K \cap \theta^\perp) = c_n \mathcal{R}\Big[\rho_K^{n-1}\Big],$$

where c_n is a positive constant depending on the dimension n only and $\mathcal{R}$ is the Radon transform on S^{n-1}, i.e.,

$$\mathcal{R}f(\theta) = \int_{S^{n-1}\cap\theta^\perp} f(\xi)d\sigma(\xi)$$

with σ being the $(n-2)$-dimensional Lebesgue measure on $S^{n-1} \cap \theta^\perp$ normalized by the condition $\sigma(S^{n-1} \cap \theta^\perp) = 1$, i.e., $\mathcal{R}1 = 1$. Thus, condition (1) can be rewritten as $h_K \mathcal{R}\Big[\rho_K^{n-1}\Big] = C$, where, due to the normalization made at the beginning of Section 4, the constant C should be the same as for the unit ball B_2^n, i.e., $C = 1$. So, we arrive at the equation

$$h_K = \Big(\mathcal{R}\Big[\rho_K^{n-1}\Big]\Big)^{-1}. \tag{6}$$

Rewriting (2) in terms of h_K and ρ_K is trickier. The right-hand side presents no problem: it is just proportional to $\left(\mathcal{R}\left[\rho_K^{n-1}\right]\right)^{n+1}$. So, the equation becomes $f_K = C\left(\mathcal{R}\left[\rho_K^{n-1}\right]\right)^{n+1}$. Due to the normalization made at the beginning of Section 4, the constant C should be the same as for the unit ball B_2^n, i.e., $C = 1$. However, f_K can be readily expressed in terms of h_K only if h_K is C^2 and we have made no such assumption.

The expression for f_K in the C^2-case can be written as $f_K = Ah_K$ where the operator A is defined as follows. For a function $h \in C^2(S^{n-1})$ denote by $H(x)$ its degree 1 homogeneous extension to the entire space (i.e., $H(x) = |x|h(\frac{x}{|x|})$ for $x \neq 0$). Let $\widehat{H} = (H_{x_i x_j})_{i,j=1}^n$ be the Hessian of H and let $\widehat{H}_j$ be the matrix obtained from H by removing the j-th row and the j-th column. Let Ah be the restriction of $\sum_{j=1}^n \det \widehat{H}_j$ to the unit sphere S^{n-1} (see [12], Corollary 2.5.3).

In the general case we shall first show that when ρ_K is close to 1, we can solve the equation

$$Ah = \left(\mathcal{R}\left[\rho_K^{n-1}\right]\right)^{n+1}$$

with h close to 1 in C^2. This h will determine a convex body L that satisfies $f_L = f_K$. By the uniqueness theorem (see [12], Theorem 8.1.1) we will then conclude that $K = L$, so $h_K = h$, i.e., the smoothness of h_K will automatically follow from equation (2), at least when $\rho_K \approx 1$. Thus, it will be possible to rewrite (2) as

$$Ah_K = \left(\mathcal{R}\left[\rho_K^{n-1}\right]\right)^{n+1}. \tag{7}$$

7. Maximal function

For $e \in S^{n-1}$, $\vartheta \in (0, \pi]$, let $S_\vartheta(e) = \{e' \in S^{n-1}, \langle e, e'\rangle \geq \cos\vartheta\}$ denote the spherical cap centered at e with spherical radius ϑ. The spherical Hardy-Littlewood maximal function is defined by

$$Mf(e) = \sup_{\vartheta\in(0,\pi]} \frac{1}{\sigma(S_\vartheta(e))} \int_{S_\vartheta(e)} |f(x)| d\sigma(x), \qquad f \in L^1(S^{n-1}),$$

where σ is the surface measure on S^{n-1} normalized by the condition $\sigma(S^{n-1}) = 1$. It is well known that M is bounded as an operator from $L^2(S^{n-1})$ to itself (see [6]).

Lemma 2. *Let K be a 2-dimensional origin-symmetric convex body and let R be a positive real number. Let $h_K = R+\omega$ be the support function of K and let $e \in S^1$ be a unit vector. Assume that $h_K(e) \leq R\cos\vartheta$ for some $\vartheta \in (0, \frac{\pi}{2})$. Denote by $e'(t)$ the unit vector situated clockwise from e and making an angle t with e. Then*

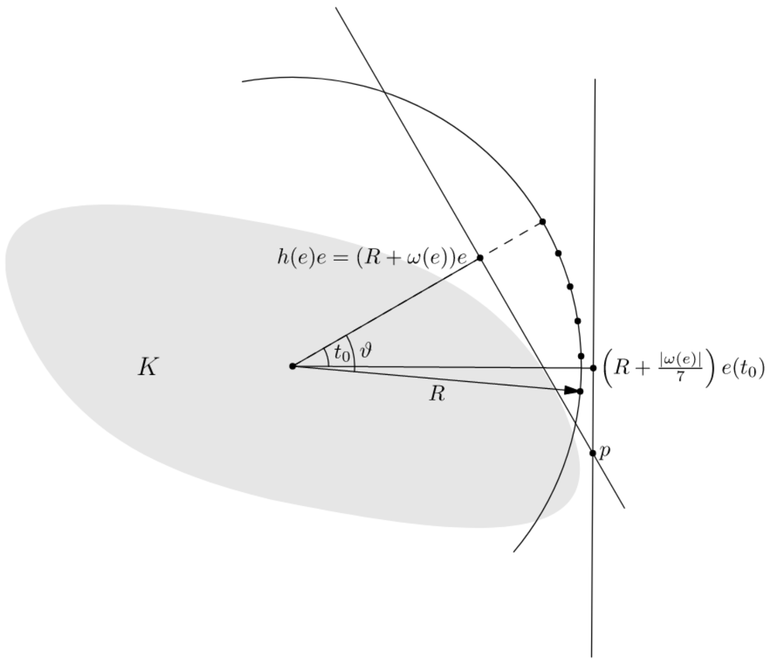

Fig. 1. The body K, the lines $\langle x, e\rangle = R + \omega(e)$, $\langle x, e'(t_0)\rangle = R + \frac{1}{7}|\omega(e)|$, and the point p.

$$|\omega(e)| \leq \frac{35}{\vartheta} \int_{\frac{\vartheta}{5}}^{\vartheta} |\omega(e'(t))| dt.$$

Proof. Note that the hypothesis $h_K(e) \leq R\cos\vartheta$ implies that $\omega(e) < 0$. If $\omega(e'(t)) \geq \frac{1}{7}|\omega(e)|$ for all $t \in [\frac{4\vartheta}{5}, \vartheta]$, the inequality clearly holds. Assume now that $\omega(e'(t_0)) < \frac{1}{7}|\omega(e)|$ for some $t_0 \in [\frac{4\vartheta}{5}, \vartheta]$. Let p be the intersection point of the lines $\langle x, e\rangle = R + \omega(e)$ and $\langle x, e'(t_0)\rangle = R + \frac{1}{7}|\omega(e)|$. Then p lies clockwise from e and, since $|p| > R$ and $\langle p, e\rangle = h_K(e) < R\cos\vartheta$, the angle α between p and e is at least ϑ (see Fig. 1). Also, since $\langle p, e\rangle = h_K(e) > 0$, we have $\alpha < \frac{\pi}{2}$.

Since K is contained entirely in the angle $\langle x, e\rangle \leq R + \omega(e)$, $\langle x, e'(t_0)\rangle \leq R + \frac{1}{7}|\omega(e)|$ with vertex p, we have $h_K(e'(t)) \leq \langle p, e'(t)\rangle = |p|\cos(\alpha - t)$ for all $t \in [0, t_0]$.

We shall now use the following elementary property of the cosine function: if γ, $\delta > 0$ and $[\gamma - \delta, \gamma + \delta] \subset [0, \frac{\pi}{2}]$, then $\cos\beta \leq \frac{3\cos(\gamma-\delta)+\cos(\gamma+\delta)}{4}$ for all $\beta \in [\gamma, \gamma + \delta]$. Indeed, since $\cos\beta \leq \cos\gamma$, it suffices to show that

$$\cos\gamma \leq \frac{3\cos(\gamma - \delta) + \cos(\gamma + \delta)}{4} = \cos\gamma\cos\delta + \frac{1}{2}\sin\gamma\sin\delta.$$

Rewriting this as $\cos\gamma(1 - \cos\delta) \leq \frac{1}{2}\sin\gamma\sin\delta$ and using the identity $1 - \cos\delta = \frac{1-\cos^2\delta}{1+\cos\delta}$ $= \frac{\sin^2\delta}{1+\cos\delta}$, we see that we need to prove that $\frac{\cos\gamma}{1+\cos\delta}\sin^2\delta \leq \frac{1}{2}\sin\gamma\sin\delta$. However, since

$0 \le \delta \le \gamma \le \frac{\pi}{2}$, we have $\cos\gamma \le \cos\delta \le 1$ and $\sin\gamma \ge \sin\delta$, so the left hand side is at most $\frac{1}{2}\sin^2\delta$ and the right hand side is at least that.

Applying this property to the interval $[\alpha - t_0, \alpha]$, i.e., with $\gamma = \alpha - \frac{t_0}{2}$, $\delta = \frac{t_0}{2}$, we conclude that

$$h_K(e'(t)) \le \frac{3}{4}\Big(R + \frac{1}{7}|\omega(e)|\Big) + \frac{1}{4}(R + \omega(e)) = R - \frac{1}{7}|\omega(e)|$$

for every $t \in [0, \frac{t_0}{2}] \supset [\frac{\vartheta}{5}, \frac{2\vartheta}{5}]$ and the conclusion of the lemma follows again. □

Corollary 1. *Let K be a convex body in $\mathbb{R}^n$ and let $R > 0$. Let $h_K = R + \omega$ be the support function of K and let $e \in S^{n-1}$ be a unit vector. Assume that $h_K(e) \le R\cos\vartheta$ for some $\vartheta \in (0, \frac{\pi}{2})$. Then*

$$|\omega(e)| \le C\,\frac{1}{\sigma(S_\vartheta(e))}\int\limits_{S_\vartheta(e)} |\omega(e')|d\sigma(e').$$

Proof. We will use the parametrization $e' = e'(t, v) \in S^{n-1}$ where t is the angle between e and e', and $v \in S^{n-1} \cap e^\perp$ is such that $e' = e\cos t + v\sin t$.

Note that $d\sigma_{n-1}(e') = c_n(\sin t)^{n-2}dtd\sigma_{n-2}(v)$. It follows from the lemma applied to the projection of K to the plane spanned by e and v that

$$|\omega(e)| \le \frac{35}{\vartheta}\int\limits_{\frac{\vartheta}{5}}^{\vartheta} |\omega(e'(t,v))|dt \le \frac{35}{\vartheta\left(\sin(\frac{\vartheta}{5})\right)^{n-2}}\int\limits_{0}^{\vartheta} |\omega(e'(t,v))|(\sin t)^{n-2}dt.$$

Integrating this inequality with respect to v and observing that $\sigma(S_\vartheta(e)) \asymp \vartheta^{n-1} \asymp \vartheta(\sin(\frac{\vartheta}{5}))^{n-2}$, we get the statement of the corollary. □

Lemma 3. *Assume that a symmetric convex body K is very close to the unit ball and $l \in \mathbb{N}$. Let $h = h_K$ and $\rho = \rho_K$ be the support and the radial functions of K respectively. Trivially, $\rho \le h$. Let $h = \sum\limits_{m=0}^{\infty} h_m$ be the decomposition of h into spherical harmonics (since h is even, only h_m with even m are not identically 0). Put $\eta = \sum\limits_{m=1}^{l} h_m$, $\nu = \sum\limits_{m=l+1}^{\infty} h_m$. Then for every $\varepsilon, l > 0$, there exists $\delta_0 = \delta_0(\varepsilon, l)$ such that whenever $\|h - 1\|_\infty \le \delta_0$, the inequality*

$$h - \rho \le \varepsilon\|\eta\|_{L^2} + CM\nu$$

holds, where C is an absolute constant and M is the spherical Hardy-Littlewood maximal function.

Proof. We have

$$\rho(e) = \inf_{\{e' \in S^{n-1}, \langle e, e' \rangle > 0\}} \frac{h(e')}{\langle e, e' \rangle}.$$

Note that the admissible range of e' can be further restricted to $|e-e'| < \delta$ with arbitrarily small $\delta > 0$, provided that δ_0 is chosen small enough. Indeed, since $h(e') \geq \frac{1-\delta_0}{1+\delta_0} h(e)$, e' can compete with e only if $\langle e, e' \rangle \geq \frac{1-\delta_0}{1+\delta_0}$, so

$$|e-e'|^2 = 2(1 - \langle e, e' \rangle) \leq \frac{4\delta_0}{1+\delta_0} < \delta^2$$

if $\delta_0 > 0$ is chosen appropriately. Now observe also that all norms on the finite-dimensional space of polynomials of degree not exceeding l on the unit sphere are equivalent, and that any semi-norm is dominated by any norm, whence

$$\|\eta\|_{C(S^{n-1})} \leq C(l)\|\eta\|_{L^2(S^{n-1})} \qquad \text{and} \qquad \|\nabla\eta\|_{C(S^{n-1})} \leq C(l)\|\eta\|_{L^2(S^{n-1})}.$$

In particular, if $|e - e''| < 2\delta$, we get

$$|\eta(e) - \eta(e'')| \leq 4\|\nabla\eta\|_{C(S^{n-1})}\delta \leq 4C(l)\delta\|\eta\|_{L^2(S^{n-1})}.$$

Let us now assume that $e' \in S^{n-1}$, with $|e - e'| < \delta$, is a competitor, so $\frac{h(e')}{\langle e, e' \rangle} \leq h(e)$. Then, if ϑ is the angle between e and e', we have $h(e') \leq h(e)\cos\vartheta$, so we can apply Corollary 1 to the vector e' with $R = h(e)$ and conclude that

$$\begin{aligned} h(e) - \frac{h(e')}{\langle e, e' \rangle} \leq h(e) - h(e') &\leq \frac{C}{\sigma(S_\vartheta(e'))} \int\limits_{S_\vartheta(e')} |h(e) - h(e'')| d\sigma(e'') \\ &\leq \frac{C'}{\sigma(S_{2\vartheta}(e))} \int\limits_{S_{2\vartheta}(e)} |h(e) - h(e'')| d\sigma(e''). \end{aligned}$$

However,

$$|h(e) - h(e'')| \leq |\eta(e) - \eta(e'')| + |\nu(e)| + |\nu(e'')|,$$

and

$$|\eta(e) - \eta(e'')| \leq 4C(l)\delta\|\eta\|_{L^2(S^{n-1})},$$

while

$$|\nu(e)| \leq M\nu(e) \qquad \text{and} \qquad \frac{1}{\sigma(S_{2\vartheta}(e))} \int\limits_{S_{2\vartheta}(e)} |\nu(e'')| d\sigma(e'') \leq M\nu(e),$$

so the desired statement follows if we choose $\delta > 0$ so that $4C'C(l)\delta < \varepsilon$. $\square$

8. Contraction

Let $\mathfrak{M}$ be a bounded linear operator on $L^2 = L^2(S^{n-1})$ such that $\mathfrak{M}$ is proportional to the identity on every space $\mathcal{H}_m$ of spherical harmonics of degree m, i.e., for some $\mu_m \in \mathbb{R}$,

$$\mathfrak{M}f = \sum_{m\geq 0} \mu_m f_m \qquad \text{where} \qquad f = \sum_{m\geq 0} f_m \qquad \text{and} \qquad f_m \in \mathcal{H}_m$$

is the spherical harmonic decomposition of $f \in L^2(S^{n-1})$. We say that $\mathfrak{M}$ is a strong contraction if

$$\max_{m\geq 0} |\mu_m| < 1, \quad \text{and} \qquad \lim_{m\to\infty} \mu_m = 0.$$

Lemma 4. *Assume that $\mathfrak{M}$ as above is a strong contraction. Then, there exists $\delta \in (0,1)$ such that for any symmetric convex body K and any $c \in (1-\delta, 1+\delta)$, the conditions*

$$1 - \delta \leq \rho_K \leq 1 + \delta, \qquad \|(h_K - h_0) - c\,\mathfrak{M}(\rho_K - r_0)\|_{L^2} \leq \delta \|\rho_K - r_0\|_{L^2},$$

imply $h_K = \rho_K = const$. Here, h_0 and r_0 are the constant terms of the spherical harmonic decomposition of h_K and ρ_K, respectively.

Proof. Fix a large l and consider the decompositions

$$h_K = h_0 + \eta + \nu \qquad \text{and} \qquad \rho_K = r_0 + \varphi + \psi,$$

where h_0, r_0 are the constant terms, η and φ are the parts corresponding to the harmonics of degrees 1 to l and ν and ψ are the parts corresponding to the harmonics of degrees greater than l.

Fix $\varepsilon > 0$. Since the projection to any sum of spaces of spherical harmonics in L^2 has norm 1, we have

$$\begin{aligned}\|\eta - c\,\mathfrak{M}\varphi\|_{L^2} \leq \|h_K - h_0 - c\,\mathfrak{M}(\rho_K - r_0)\|_{L^2} \leq \\ \delta\|\rho_K - r_0\|_{L^2} \leq \delta(\|\varphi\|_{L^2} + \|\psi\|_{L^2}).\end{aligned} \tag{8}$$

Similarly,

$$\|\nu - c\,\mathfrak{M}\psi\|_{L^2} \leq \delta(\|\varphi\|_{L^2} + \|\psi\|_{L^2}). \tag{9}$$

From (9) we obtain that

$$\begin{aligned}\|\nu\|_{L^2} \leq c\,\|\mathfrak{M}\psi\|_{L^2} + \delta(\|\varphi\|_{L^2} + \|\psi\|_{L^2}) \leq \\ (1+\delta)(\max_{m>l} |\mu_m|)\|\psi\|_{L^2} + \delta(\|\varphi\|_{L^2} + \|\psi\|_{L^2}) \leq \varepsilon(\|\varphi\|_{L^2} + \|\psi\|_{L^2})\end{aligned} \tag{10}$$

if l is large enough (recall that $\lim_{m\to\infty} \mu_m = 0$) and δ is small enough. The same computation for η, using in this case that $\max_{m\geq 0} |\mu_m| < 1$, yields

$$\|\eta\|_{L^2} \leq (1+2\delta)(\|\varphi\|_{L^2} + \|\psi\|_{L^2}). \tag{11}$$

On the other hand, by Lemma 3 and the boundedness of the maximal function in L^2, we have

$$\|h_K - \rho_K\|_{L^2} \leq \varepsilon\|\eta\|_{L^2} + C\|\nu\|_{L^2},$$

which implies

$$\|\eta - \varphi\|_{L^2} \leq \varepsilon\|\eta\|_{L^2} + C\|\nu\|_{L^2} \quad \text{and} \quad \|\nu - \psi\|_{L^2} \leq \varepsilon\|\eta\|_{L^2} + C\|\nu\|_{L^2}. \tag{12}$$

Combining (8), (9), (10), (11), and (12), we obtain

$$\begin{aligned} &\|\varphi - c\,\mathfrak{M}\varphi\|_{L^2} + \|\psi - c\,\mathfrak{M}\psi\|_{L^2} \\ &\leq \|\varphi - \eta\|_{L^2} + \|\eta - c\,\mathfrak{M}\varphi\|_{L^2} + \|\psi - \nu\|_{L^2} + \|\nu - c\,\mathfrak{M}\psi\|_{L^2} \\ &\leq C(\delta+\varepsilon)(\|\varphi\|_{L^2} + \|\psi\|_{L^2}). \end{aligned}$$

On the other hand, for any function $\chi \in L^2(S^{n-1})$, we have

$$\|\chi - c\,\mathfrak{M}\chi\|_{L^2} \geq (1 - (1+\delta)\max_{m\geq 0} |\mu_m|)\|\chi\|_{L^2},$$

so we can conclude that $\varphi = 0$, $\psi = 0$ if $C(\delta+\varepsilon) < 1 - (1+\delta)\max_{m\geq 0} |\mu_m|$. $\square$

Remark 1. Note that $(h_K - h_0) - c\,\mathfrak{M}(\rho_K - r_0)$ is orthogonal to constants and, therefore, its L^2-norm does not exceed $\|(h_K - \lambda) - c\,\mathfrak{M}(\rho_K - r_0)\|_{L^2}$ for any $\lambda \in \mathbb{R}$. Thus, to verify the conditions of the lemma it suffices to check that

$$\|(h_K - \lambda) - c\,\mathfrak{M}(\rho_K - r_0)\|_{L^2} \leq \delta\|\rho_K - r_0\|_{L^2}$$

with any $\lambda \in \mathbb{R}$ of our choice.

9. Properties of the function $(\mathcal{R}[\rho_K^\alpha])^\beta$ when ρ_K is close to 1

Let K be a symmetric convex body in the isotropic position such that $1-\delta \leq \rho_K \leq 1+\delta$ for some small $\delta > 0$. Let $\alpha, \beta \in \mathbb{R}$. We want to derive several useful properties of the function $(\mathcal{R}[\rho_K^\alpha])^\beta$.

The first observation is that ρ_K is Lipschitz with Lipschitz constant $5\sqrt{\delta}$. Indeed, let $x, y \in S^{n-1}$. If $|x-y| \geq \frac{\sqrt{\delta}}{2}$, then we have

$$|\rho_K(x) - \rho_K(y)| \leq 2\delta \leq 4\sqrt{\delta}|x-y|,$$

so we may assume that $0 < |x-y| < \frac{\sqrt{\delta}}{2}$. Without loss of generality, $\rho_K(x) \geq \rho_K(y)$. Let us denote $X = \rho_K(x)x$, $Y = \rho_K(y)y$, where $X, Y \in \partial K$. By the convexity of K, every point on the line $Y - t(X-Y)$ with $t \geq 0$ lies outside K and, therefore, outside $(1-\delta)B_2^n$ as well. Hence,

$$(1-\delta)^2 \leq |Y - t(X-Y)|^2 = |Y|^2 - 2t\langle X-Y, Y\rangle + t^2|X-Y|^2.$$

Since $|Y|^2 \leq (1+\delta)^2$, we conclude that, for all $t \geq 0$,

$$2t\langle X-Y, Y\rangle - t^2|X-Y|^2 \leq 4\delta. \tag{13}$$

From (13) it follows that

$$\langle X-Y, Y\rangle \leq 2\sqrt{\delta}|X-Y|. \tag{14}$$

Indeed, if $\langle X-Y, Y\rangle \leq 0$, the inequality is obvious. Otherwise, we can plug $t = \frac{\langle X-Y,Y\rangle}{|X-Y|^2}$ into (13), obtaining $\frac{\langle X-Y,Y\rangle^2}{|X-Y|^2} \leq 4\delta$, which is equivalent to (14). Now, equation (14) can be rewritten as

$$|X||Y|\langle x, y\rangle - |Y|^2 \leq 2\sqrt{\delta}|X-Y|,$$

or, equivalently,

$$|Y|\left(|X| - |Y|\right) \leq 2\sqrt{\delta}|X-Y| + |X||Y|(1 - \langle x, y\rangle).$$

Observe that $1 - \langle x, y\rangle = \frac{1}{2}|x-y|^2 \leq \frac{\sqrt{\delta}}{4}|x-y|$, while $|X-Y| \leq |X| - |Y| + |Y||x-y|$. Hence,

$$|X| - |Y| \leq \frac{2\sqrt{\delta}}{|Y|}\left(|X| - |Y|\right) + \left(2 + \frac{|X|}{4}\right)\sqrt{\delta}|x-y|.$$

Now, if $\delta \in (0, 1/25)$,

$$\frac{2\sqrt{\delta}}{|Y|} \leq \frac{2\sqrt{\delta}}{1-\delta} \leq \frac{2/5}{1-1/25} < \frac{1}{2},$$

and we conclude that

$$|X| - |Y| \leq 2\left(2 + \frac{|X|}{4}\right)\sqrt{\delta}|x-y| \leq \left(4 + \frac{1+\delta}{2}\right)\sqrt{\delta}|x-y| \leq 5\sqrt{\delta}|x-y|,$$

as required.

Since the mapping $t \mapsto t^p$ is Lipschitz on any compact subset of $(0, +\infty)$ and the Radon transform does not increase the Lipschitz constant of the function, we immediately conclude that $(\mathcal{R}[\rho_K^\alpha])^\beta$ has Lipschitz constant at most $C_{\alpha,\beta}\sqrt{\delta}$.

Let now $r_0 = \int_{S^{n-1}} \rho_K d\sigma$ be the mean value of ρ_K on the unit sphere. Clearly, $|r_0 - 1| \le \delta$, so $|\rho_K - r_0| \le 2\delta$ and, thereby, $\mathcal{R}|\rho_K - r_0| \le 2\delta$ as well. Now, using the fact that $t \mapsto t^p$ is C^2 on any compact subset of $(0, +\infty)$ and linearizing, we successively derive that

$$\begin{aligned}
|\rho_K^\alpha - (r_0^\alpha + \alpha r_0^{\alpha-1}(\rho_K - r_0))| &\le C_\alpha \delta |\rho_K - r_0|, \\
|\mathcal{R}[\rho_K^\alpha] - (r_0^\alpha + \alpha r_0^{\alpha-1}\mathcal{R}(\rho_K - r_0))| &\le C_\alpha\, \delta \mathcal{R}|\rho_K - r_0|, \\
|(\mathcal{R}[\rho_K^\alpha])^\beta - (r_0^\alpha + \alpha r_0^{\alpha-1}\mathcal{R}(\rho_K - r_0))^\beta| &\le C_{\alpha,\beta}\, \delta \mathcal{R}|\rho_K - r_0|, \\
|(\mathcal{R}[\rho_K^\alpha])^\beta - (r_0^{\alpha\beta} + \alpha\beta r_0^{\alpha\beta-1}\mathcal{R}(\rho_K - r_0))| &\le C_{\alpha,\beta}\, \delta \mathcal{R}|\rho_K - r_0|.
\end{aligned}$$

Thus, $(\mathcal{R}[\rho_K^\alpha])^\beta = r_0^{\alpha\beta} + \gamma$, where

$$|\gamma - \alpha\beta r_0^{\alpha\beta-1}\mathcal{R}(\rho_K - r_0)| \le C_{\alpha,\beta}\, \delta \mathcal{R}|\rho_K - r_0|.$$

In particular, we conclude that $|\gamma| \le C_{\alpha,\beta}\, \delta$, which, together with the above observation about the Lipschitz constant, implies that

$$\|\gamma\|_{C^{\frac{1}{2}}} = \max_{S^{n-1}} |\gamma| + \sup_{x,y \in S^{n-1}, x \ne y} \frac{|\gamma(x) - \gamma(y)|}{|x - y|^{\frac{1}{2}}} \le C_{\alpha,\beta}\sqrt{\delta}.$$

Now let $\rho_K - r_0 = Y_2 + Y_4 + \dots$ be the spherical harmonic decomposition of $\rho_K - r_0$. It follows from the definition of the isotropic position that

$$0 = \int_K p(x)dx = c_n \int_{S^{n-1}} \rho_K^{n+2}(x)p(x)d\sigma(x)$$

for all quadratic polynomials $p(x) = \sum_{i,j} a_{ij}x_i x_j$ with $\sum_{i=1}^n a_{ii} = 0$. In other words, ρ_K^{n+2} has no second order term in its spherical harmonic decomposition.

On the other hand,

$$|\rho_K^{n+2} - (r_0^{n+2} + (n+2)r_0^{n+1}(\rho_K - r_0))| \le C\delta|\rho_K - r_0|.$$

Taking the second order component in the spherical harmonic decomposition of the expression under the absolute value sign on the left hand side, we get

$$(n+2)r_0^{n+1}\|Y_2\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})},$$

so

$$\|Y_2\|_{L^2(S^{n-1})} \le C'\delta\|\rho_K - r_0\|_{L^2(S^{n-1})}.$$

10. A solution to the fifth Busemann-Petty problem in a small neighborhood of the Euclidean ball

Recall that for the fifth Busemann-Petty problem we have the equation $h_K = (\mathcal{R}[\rho_K^{n-1}])^{-1}$. By the results of the previous section, the right hand side can be written as

$$r_0^{-n+1} - (n-1)r_0^{-n}\mathcal{R}(\rho_K - r_0) + \gamma',$$

where $|\gamma'| \le C\delta\mathcal{R}|\rho_K - r_0|$, so $\|\gamma'\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})}$.

Let $\mathfrak{M}$ be the linear operator that maps every m-th order spherical harmonic Z_m to

$$-(n-1)\mathcal{R}Z_m = -(n-1)(-1)^{\frac{m}{2}}\frac{1\cdot 3\cdot\ldots\cdot(m-1)}{(n-1)(n+1)\cdot\ldots\cdot(n+m-3)}Z_m$$

for even $m \ge 4$ and to 0 for other m. Then $\mathfrak{M}$ is a strong contraction and

$$\|(h_K - r_0^{-n+1}) - r_0^{-n}\mathfrak{M}(\rho_K - r_0)\|_{L^2(S^{n-1})} \le$$
$$r_0^{-n}\|Y_2\|_{L^2(S^{n-1})} + \|\gamma'\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})},$$

so Lemma 4 and Remark 1 yield $h_K = \rho_K = \mathit{const}$, i.e., K is a ball, provided that δ is small enough.

11. A solution to the eighth Busemann-Petty problem in a small neighborhood of the Euclidean ball

We now turn to the equation $Ah_K = (\mathcal{R}[\rho_K^{n-1}])^{n+1}$ (see Section 6). Below we will use several standard results about A and the Laplace operator which, for completeness, are proven in the Appendices.

By the results of Section 9, $(\mathcal{R}[\rho_K^{n-1}])^{n+1}$ can be rewritten as $r_0^{(n-1)(n+1)} + \gamma$, where $\|\gamma\|_{C^{\frac{1}{2}}} \le C\sqrt{\delta}$ and

$$\gamma = (n-1)(n+1)r_0^{(n-1)(n+1)-1}\mathcal{R}(\rho_K - r_0) + \gamma',$$
$$\|\gamma'\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})}.$$

Then

$$A\frac{h_K}{r_0^{n+1}} = 1 + r_0^{-(n-1)(n+1)}\gamma$$

and, provided that $\delta > 0$ is small enough, we can apply Lemma 5 (see Appendix II) and the uniqueness theorem (see [12], Theorem 8.1.1) to obtain

$$\frac{h_K}{r_0^{n+1}} = 1 + \varphi' + \varphi'',$$

where

$$\widetilde{\Delta}\varphi' = r_0^{-(n-1)(n+1)}\gamma \tag{15}$$

(see Appendix II for the definition of $\widetilde{\Delta}$) and

$$\|\varphi''\|_{L^2(S^{n-1})} \le \varepsilon r_0^{-(n-1)(n+1)}\|\gamma\|_{L^2(S^{n-1})} \le C\varepsilon\|\rho_K - r_0\|_{L^2(S^{n-1})}$$

with as small $\varepsilon > 0$ as we want.

Furthermore, the solution of equation (15) splits into $\varphi_1' + \varphi_2'$ where φ_1' solves

$$\widetilde{\Delta}\varphi_1' = (n-1)(n+1)r_0^{-1}\mathcal{R}(\rho_K - r_0),$$

and φ_2' solves $\widetilde{\Delta}\varphi_2' = r_0^{-(n-1)(n+1)}\gamma'$.

The norm of φ_2' can be estimated immediately:

$$\|\varphi_2'\|_{L^2(S^{n-1})} \le C\|\gamma'\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})}.$$

As to φ_1', it is equal to (see the end of Appendix I)

$$r_0^{-1}\sum_{\substack{m\ge 2\\ m\,\text{even}}} \mu_m Y_m,$$

where $\rho_K = r_0 + \sum_{\substack{m\ge 2\\ m\,\text{even}}} Y_m$ is the spherical harmonic decomposition of ρ_K and

$$\mu_m = \frac{(n-1)(n+1)}{(1-m)(m+n-1)}(-1)^{\frac{m}{2}}\frac{1\cdot 3\cdot \ldots \cdot (m-1)}{(n-1)(n+1)\cdot \ldots \cdot (n+m-3)},$$

so $\mu_2 = 1$ and $|\mu_m| < 1$ for $m \ge 4$, $\mu_m \to 0$ as $m \to \infty$. Since

$$\|Y_2\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})},$$

we conclude that

$$\|\varphi_1' - r_0^{-1}\mathfrak{M}(\rho_K - r_0)\|_{L^2(S^{n-1})} \le C\delta\|\rho_K - r_0\|_{L^2(S^{n-1})},$$

with the strong contraction $\mathfrak{M}$ given by $Z_m \mapsto \mu_m Z_m$, m even, $m \ge 4$; $Z_m \mapsto 0$ for all other m.

Putting all these estimates together, we conclude that

$$\|(h_K - r_0^{n+1}) - r_0^n\mathfrak{M}(\rho_K - r_0)\|_{L^2(S^{n-1})} \le \varepsilon\|\rho_K - r_0\|_{L^2(S^{n-1})},$$

with as small $\varepsilon > 0$ as we want, provided that $\delta > 0$ is small enough. Now we can apply Lemma 4 and Remark 1 again to conclude that $h_K = \rho_K = const$, so K is a ball.

12. Appendix I. Solving the Laplace equation

Below we shall use the following notation. For a function $f : S^{n-1} \to \mathbb{R}$ and $\alpha \in (0,1)$, we shall denote

$$\|f\|_{C^\alpha} = \|f\|_{C^\alpha(S^{n-1})} = \max_{S^{n-1}} |f| + \sup_{x,y\in S^{n-1},\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

$$\|f\|_{C^{2+\alpha}} = \|f\|_{C^{2+\alpha}(S^{n-1})} =$$

$$\max_{S^{n-1}} |f| + \max_{x\in S^{n-1},\, i=1,\dots,n} |F_{x_i}(x)| + \max_{i,j=1,\dots,n} \|F_{x_i x_j}\|_{C^\alpha(S^{n-1})},$$

where $F(x) = |x| f(\frac{x}{|x|})$ is the 1-homogeneous extension of f to $\mathbb{R}^n \setminus \{0\}$ (we assume that it is at least C^2 in $\mathbb{R}^n \setminus \{0\}$).

Let $g : S^{n-1} \to \mathbb{R}$ be an even C^α function on the unit sphere S^{n-1} with some $\alpha \in (0,1)$. Let $\mathcal{G}$ be the (-1)-homogeneous extension of g to $\mathbb{R}^n \setminus \{0\}$, i.e., $\mathcal{G}(x) = |x|^{-1} g(\frac{x}{|x|})$ for $x \neq 0$. We will show that there exists a unique 1-homogeneous even function $F : \mathbb{R}^n \to \mathbb{R}$ of class L^1_{loc} such that $\Delta F = \mathcal{G}$ in $\mathbb{R}^n$ in the sense of generalized functions. Moreover, $F \in C^{2+\alpha}(S^{n-1})$ and for all $i,j = 1,\dots,n$, we have

$$\|F_{x_i x_j}\|_{L^2(S^{n-1})} \le C\|g\|_{L^2(S^{n-1})}, \qquad \|F\|_{C^{2+\alpha}(S^{n-1})} \le C\|g\|_{C^\alpha(S^{n-1})}, \tag{16}$$

with some $C = C(n,\alpha) > 0$.

12.1. Uniqueness

If we have two even 1-homogeneous functions F_1, F_2 such that $\Delta F_1 = \Delta F_2 = \mathcal{G}$ in $\mathbb{R}^n$, then $F_1 - F_2$ is an even 1-homogeneous harmonic function, but the only such function is 0.

12.2. Existence

Now we will show that the function F defined by

$$F(x) = c_n \int\limits_{\mathbb{R}^n} \Big[\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}}\Big] \mathcal{G}(y)dy$$

is a well-defined 1-homogeneous function on $\mathbb{R}^n$ satisfying $\Delta F = \mathcal{G}$ and estimates (16). Here, c_n is chosen so that $\Delta \frac{c_n}{|x|^{n-2}} = \delta_0$ (the Dirac delta measure) in the sense of generalized functions, and the integral is understood as $\lim\limits_{R\to\infty} \int\limits_{B(0,R)}$.

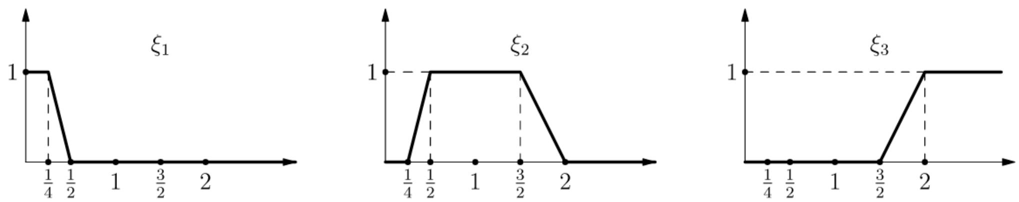

Fig. 2. The functions ξ_i, $i = 1, 2, 3$.

In order to show the convergence of the integral, we note that

$$\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} = (n-2)\frac{\langle x, y\rangle}{|y|^n} + O\Big(\frac{1}{|y|^n}\Big)$$

as $y \to \infty$ uniformly on compact sets in x.

Since $\mathcal{G}$ is even, the integral of $\frac{\langle x,y\rangle}{|y|^n}\mathcal{G}(y)$ over each sphere centered at the origin vanishes. Since $\mathcal{G}$ is (-1)-homogeneous, we have $\frac{1}{|y|^n}\mathcal{G}(y) = O(\frac{\|g\|_C}{|y|^{n+1}})$ as $y \to \infty$, which is integrable at ∞.

The singularities at $x \in S^{n-1}$ and 0 are of degrees $-(n-2)$ and $-(n-1)$ respectively, so the local integrability there presents no problem either, and we get the estimate

$$\|F\|_{C(S^{n-1})} \leq C\|g\|_{C(S^{n-1})}.$$

The change of variable $y \mapsto -y$ and the identity $\mathcal{G}(y) = \mathcal{G}(-y)$ imply that F is even.

To show the 1-homogeneity of F, take $t > 0$ and apply the change of variable $y \mapsto ty$ to write

$$\begin{aligned} F(tx) = c_n \lim_{R\to\infty} \int\limits_{B(0,R)} \Big(\frac{1}{|tx-y|^{n-2}} - \frac{1}{|y|^{n-2}}\Big)\mathcal{G}(y)dy = \\ c_n \lim_{R\to\infty} \int\limits_{B(0,\frac{R}{t})} \Big(\frac{1}{|tx-ty|^{n-2}} - \frac{1}{|ty|^{n-2}}\Big)\mathcal{G}(ty)d(ty) = \\ c_n\, t \lim_{R\to\infty} \int\limits_{B(0,\frac{R}{t})} \Big(\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}}\Big)\mathcal{G}(y)dy = tF(x), \end{aligned}$$

(we used that $\frac{1}{|tz|^{n-2}} = \frac{1}{t^{n-2}}\frac{1}{|z|^{n-2}}$, $\mathcal{G}(ty) = t^{-1}\mathcal{G}(y)$, and $d(ty) = t^n dy$).

To estimate $\|F\|_{C^{2+\alpha}(S^{n-1})}$, we split the integral defining F into 3 parts. Let ξ_1, ξ_2, $\xi_3 : [0, +\infty) \to [0, 1]$ be as on Fig. 2, so ξ_i are Lipschitz with constant 4, and $\xi_1 + \xi_2 + \xi_3 = 1$.

Put $\mathcal{G}_i(x) = \mathcal{G}(x)\xi_i(|x|)$ and

$$F_i(x) = c_n \int\limits_{\mathbb{R}^n} \Big[\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}}\Big]\mathcal{G}_i(y)dy,$$

so $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3$ and $F = F_1 + F_2 + F_3$.

Our first observation is that $\mathcal{G}_2(y)$ is an α-Hölder, compactly supported function on $\mathbb{R}^n$, with C^α-norm bounded by $C\|g\|_{C^\alpha(S^{n-1})}$.

Indeed, we clearly have $\max\limits_{\mathbb{R}^n} |\mathcal{G}_2| \le 4 \max\limits_{S^{n-1}} |g|$. On the other hand,

$$|\mathcal{G}_2(x) - \mathcal{G}_2(y)| = \Big|\xi_2(|x|)|x|^{-1} g\Big(\frac{x}{|x|}\Big) - \xi_2(|y|)|y|^{-1} g\Big(\frac{y}{|y|}\Big)\Big|.$$

Since $\widetilde{\xi}_2(t) = \xi_2(t)t^{-1}$ is a compactly supported Lipschitz function on $[0, +\infty)$, it is also α-Hölder for any $\alpha \in (0, 1)$, i.e.,

$$|\widetilde{\xi}_2(t) - \widetilde{\xi}_2(s)| \le C|t-s|^\alpha \quad \text{for all} \quad t, s \ge 0.$$

Thus, if $x, y \in \overline{B(0,2)} \setminus B(0, \frac{1}{4})$, then

$$\begin{gathered}
\left|\widetilde{\xi}_2(|x|) g\left(\frac{x}{|x|}\right) - \widetilde{\xi}_2(|y|) g\left(\frac{y}{|y|}\right)\right| \le \\
|\widetilde{\xi}_2(|x|) - \widetilde{\xi}_2(|y|)| \left|g\left(\frac{x}{|x|}\right)\right| + |\widetilde{\xi}_2(|y|)| \left|g\left(\frac{x}{|x|}\right) - g\left(\frac{y}{|y|}\right)\right| \le \\
C\left||x| - |y|\right|^\alpha \max_{S^{n-1}} |g| + 4\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^\alpha \|g\|_{C^\alpha(S^{n-1})} \le \\
C\,\|g\|_{C^\alpha(S^{n-1})}\Big(|x-y|^\alpha + \Big|\frac{x}{|x|} - \frac{y}{|y|}\Big|^\alpha\Big) \le C\,\|g\|_{C^\alpha(S^{n-1})} |x-y|^\alpha,
\end{gathered}$$

because the mapping $x \mapsto \frac{x}{|x|}$ is C^1 and, thereby, Lipschitz on $\overline{B(0,2)} \setminus B(0, \frac{1}{4})$.

If $x, y \notin \overline{B(0,2)} \setminus B(0, \frac{1}{4})$, then $\mathcal{G}_2(x) = \mathcal{G}_2(y) = 0$, so the inequality

$$|\mathcal{G}_2(x) - \mathcal{G}_2(y)| \le C\,\|g\|_{C^\alpha(S^{n-1})} |x-y|^\alpha$$

holds trivially. Finally, if $x \in \overline{B(0,2)} \setminus B(0, \frac{1}{4})$ but $y \notin \overline{B(0,2)} \setminus B(0, \frac{1}{4})$, then the segment $[x, y]$ intersects the boundary of $\overline{B(0,2)} \setminus B(0, \frac{1}{4})$ at some point y', so $\mathcal{G}_2(y) = \mathcal{G}_2(y') = 0$ and

$$\begin{gathered}
|\mathcal{G}_2(x) - \mathcal{G}_2(y)| = |\mathcal{G}_2(x) - \mathcal{G}_2(y')| \le \\
C\,\|g\|_{C^\alpha(S^{n-1})} |x-y'|^\alpha \le C\,\|g\|_{C^\alpha(S^{n-1})} |x-y|^\alpha.
\end{gathered}$$

The functions $\mathcal{G}_1$ and $\mathcal{G}_3$ are supported on $\overline{B(0, \frac{1}{2})}$ and $\mathbb{R}^n \setminus B(0, \frac{3}{2})$, respectively, and satisfy the bound

$$|\mathcal{G}_1(y)|,\, |\mathcal{G}_3(y)| \le \frac{1}{|y|} \max_{S^{n-1}} |g|.$$

Now we are ready to estimate $\|F\|_{C^{2+\alpha}(S^{n-1})}$. Consider x with $\frac{3}{4} \le |x| \le \frac{5}{4}$. Note that $x \mapsto \frac{1}{|x-y|^{n-2}}$ is a C^3-function (in x) in this domain with uniformly bounded (in y) C^3-norm as long as $y \in \overline{B(0,\frac{1}{2})}$. Hence, $F_1 \in C^3(\overline{B(0,\frac{5}{4})} \setminus B(0,\frac{3}{4}))$ and

$$\|F_1\|_{C^3(\overline{B(0,\frac{5}{4})}\setminus B(0,\frac{3}{4}))} \le C\|g\|_{L^1(S^{n-1})}$$

(the constant term $\int\limits_{\mathbb{R}^n} \frac{1}{|y|^{n-2}}\mathcal{G}_1(y)dy$ is also bounded by $C\|g\|_{L^1(S^{n-1})}$).

To estimate F_3, note that for $|x| \le \frac{5}{4}$ and $|y| \ge \frac{3}{2}$, we have

$$\Big|\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}} - (n-2)\frac{\langle x,y\rangle}{|y|^n}\Big| \le \frac{C}{|y|^n},$$

$$\Big|\frac{\partial}{\partial x_i}\frac{1}{|x-y|^{n-2}} - (n-2)\frac{y_i}{|y|^n}\Big| \le \frac{C}{|y|^n},$$

$$\Big|\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|x-y|^{n-2}}\Big| \le \frac{C}{|y|^n},$$

and

$$\Big|\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\frac{1}{|x-y|^{n-2}}\Big| \le \frac{C}{|y|^{n+1}}.$$

Since $y \mapsto \frac{\langle x,y\rangle}{|y|^n}$ and $y \mapsto \frac{y_i}{|y|^n}$ are odd functions, their integrals against the even function $\mathcal{G}_3(y)$ over any sphere centered at the origin are 0 and, therefore,

$$|F_3|, |\nabla F_3|, |\nabla^2 F_3| \le C \int\limits_{\mathbb{R}^n\setminus B(0,\frac{3}{2})} |y|^{-n}|\mathcal{G}_3(y)|dy \le C\|g\|_{L^1(S^{n-1})}$$

and

$$|\nabla^3 F_3| \le C \int\limits_{\mathbb{R}^n\setminus B(0,\frac{3}{2})} |y|^{-n-1}|\mathcal{G}_3(y)|dy \le C\|g\|_{L^1(S^{n-1})},$$

so

$$\|F_3\|_{C^3(\overline{B(0,\frac{5}{4})})} \le C\|g\|_{L^1(S^{n-1})}.$$

It remains to estimate F_2. We clearly have

$$|F_2(x)| \le C\|g\|_{C(S^{n-1})} \int\limits_{\overline{B(0,2)}\setminus B(0,\frac{1}{4})} \Big|\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}}\Big|dy \le C\|g\|_{C(S^{n-1})}$$

and

$$|\nabla F_2(x)| \le C\|g\|_{C(S^{n-1})} \int\limits_{\overline{B(0,2)}\setminus B(0,\frac{1}{4})} \frac{1}{|x-y|^{n-1}} dy \le C\|g\|_{C(S^{n-1})}.$$

As for $(F_2)_{x_ix_j}$, these partial derivatives are images of $\mathcal{G}_2$ under certain Calderón-Zygmund singular integral operators (see [5], Lemma 4.4 and Theorem 9.9), so, since $\mathcal{G}_2 \in C^\alpha(\mathbb{R}^n)$ and has fixed compact support, we obtain that

$$\|(F_2)_{x_ix_j}\|_{C^\alpha(\mathbb{R}^n)} \le C\|\mathcal{G}_2\|_{C^\alpha(\mathbb{R}^n)} \le C\|g\|_{C^\alpha(S^{n-1})}$$

and

$$\|(F_2)_{x_ix_j}\|_{L^2(\mathbb{R}^n)} \le C\|\mathcal{G}_2\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{L^2(S^{n-1})}.$$

The final conclusion is that

$$\|F\|_{C^{2+\alpha}(S^{n-1})} \le C\|F\|_{C^{2+\alpha}(\overline{B(0,\frac{5}{4})}\setminus B(0,\frac{3}{4}))} \le C\|g\|_{C^\alpha(S^{n-1})}$$

and

$$\|(F_2)_{x_ix_j}\|_{L^2(S^{n-1})} \le C\|(F_2)_{x_ix_j}\|_{L^2(\overline{B(0,\frac{5}{4})}\setminus B(0,\frac{3}{4}))} \le C\|g\|_{L^2(S^{n-1})}$$

(we used the (-1)-homogeneity of $(F_2)_{x_ix_j}$ here).

The desired equality $\Delta F = \mathcal{G}$ follows from the fact that the mapping $x \mapsto \frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2}}$ is harmonic in x for $|x| \le 1$, $|y| \ge \frac{3}{2}$. This implies that $\Delta F_3 = 0$ in $B(0,1)$, while $F_1 + F_2$ differs by a constant from the classical Newton potential of the compactly supported L^1 function $\mathcal{G}_1 + \mathcal{G}_2 = \mathcal{G}$ in $B(0,1)$. Hence, $\Delta F = \mathcal{G}$ in $B(0,1)$, and this identity extends to $\mathbb{R}^n$ by homogeneity.

We shall also need the relation between the spherical harmonic decompositions of $F|_{S^{n-1}}$ and g. To this end, we will start with the following computation. Let P_m be a homogeneous harmonic polynomial of degree m, so that $Y_m = P_m|_{S^{n-1}}$ is a spherical harmonic of degree m. The 1-homogeneous extension of Y_m is $\widetilde{Y}_m(x) = |x|^{1-m}P_m(x)$. Then

$$\Delta\widetilde{Y}_m(x) = \Delta(|x|^{1-m})P_m(x) + 2\langle\nabla(|x|^{1-m}), \nabla P_m(x)\rangle =$$
$$(1-m)(-m-1+n)|x|^{-m-1}P_m(x) + 2(1-m)|x|^{-m}\frac{\partial}{\partial r}P_m(x) =$$
$$|x|^{-1-m}(1-m)(-m-1+n+2m)P_m(x) =$$
$$(1-m)(m+n-1)|x|^{-1-m}P_m(x).$$

Thus, if $g \in L^2(S^{n-1})$ and $g = \sum\limits_{\substack{m\ge 0 \\ m \text{ even}}} Y_m$ on S^{n-1}, the series

$$F = \sum_{\substack{m\geq 0 \\ m \text{ even}}} \frac{1}{(1-m)(m+n-1)} \widetilde{Y}_m$$

converges in L^2_{loc} (the series is orthogonal on every ball $B(0,R)$ and $\|\widetilde{Y}_m\|_{L^2(B(0,R))} \leq C_R\|Y_m\|_{L^2(S^{n-1})}$) and formally solves $\Delta F = \mathcal{G}$. To show that it is a true solution, it suffices to observe that we have $\Delta F_{(l)} = \mathcal{G}_{(l)}$ for the partial sums $F_{(l)}$ and $\mathcal{G}_{(l)}$ of the corresponding series and $F_{(l)} \to F$ in L^2_{loc}, $\mathcal{G}_{(l)} \to \mathcal{G}$ in L^1_{loc} as $l \to \infty$. Thus, $\Delta F = \mathcal{G}$ in the sense of generalized functions. If $g \in C^\alpha(S^{n-1})$, then, by the uniqueness part, this solution has to coincide with the explicit solution constructed above, so the spherical harmonic decomposition of $F|_{S^{n-1}}$ is $\sum_{\substack{m\geq 0 \\ m \text{ even}}} \frac{1}{(1-m)(m+n-1)} Y_m$. In particular, the decomposition implies that

$$\|F\|_{L^2(S^{n-1})} \leq \|g\|_{L^2(S^{n-1})}.$$

13. Appendix II. Solution of Monge-Ampere equation

For a function $f : S^{n-1} \to \mathbb{R}$, we denote by F its 1-homogeneous extension to $\mathbb{R}^n$. By Af we will denote the restriction of $\sum_{k=1}^{n} \det \widehat{F}_k$ to the unit sphere where $\widehat{F}_k$ is the matrix obtained from the Hessian $\widehat{F} = (F_{x_i x_j})_{i,j=1}^n$ by deleting the k-th row and the k-th column.

We now turn to the solution of the equation $Af = g$ where g is close to 1. Note that $A1 = 1$. Indeed, since A commutes with the rotations of the sphere, we can check this identity at the point $(1, 0, \dots, 0)$. The 1-homogeneous extension of 1 is $|x|$, so the Hessian is $\left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right)_{i,j=1}^n$, which at the point $(1, 0, \dots, 0)$ turns into

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

The rotation invariance also allows us to compute the linear part of Af (meaning the linear terms in $\Phi_{x_i x_j}$) for $f = 1 + \varphi$, where Φ is the 1-homogeneous extension of φ. Again, computing the Hessian at $(1, 0, \dots, 0)$, we get

$$\widehat{F} = \begin{bmatrix} \Phi_{x_1 x_1} & \Phi_{x_1 x_2} & \dots & \Phi_{x_1 x_n} \\ \Phi_{x_2 x_1} & 1 + \Phi_{x_2 x_2} & \dots & \Phi_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{x_n x_1} & \Phi_{x_n x_2} & \dots & 1 + \Phi_{x_n x_n} \end{bmatrix},$$

so

$$\sum_{i=1}^{n} \det \widehat{F}_i = \det \widehat{F}_1 + \sum_{i=2}^{n} \det \widehat{F}_i =$$

$$1 + \sum_{i=2}^{n} \Phi_{x_i x_i} + (n-1)\Phi_{x_1 x_1} + P(\Phi),$$

where $P(\Phi)$ is some linear combination of products of two or more second partial derivatives of Φ. Note now that, since Φ is 1-homogeneous, the mapping $t \mapsto \Phi(t, 0, \dots, 0)$ is linear and, thereby, $\Phi_{x_1x_1}(1, 0, \dots, 0) = 0$. Thus we can just as well write $\sum\limits_{i=2}^{n} \Phi_{x_i x_i} + (n-1)\Phi_{x_1x_1}$ at $(1, 0, \dots, 0)$ as $\Delta\Phi(1, 0, \dots, 0)$. However, $\Delta\Phi$ also commutes with rotations, so we have the identity

$$\sum_{i=1}^{n} \det \widehat{F}_i = 1 + \Delta\Phi + P(\Phi)$$

in general, though $P(\Phi)$ will now be a sum of products of at least two second partial derivatives of Φ and some fixed functions of x that are smooth near the unit sphere.

Using identities of the type

$$a_1a_2 \dots a_m - b_1b_2 \dots b_m = (a_1 - b_1)a_2 \dots a_m +$$

$$b_1(a_2 - b_2)a_3 \dots a_m + \dots + b_1 \dots b_{m-2}(a_{m-1} - b_{m-1})a_m + b_1 \dots b_{m-1}(a_m - b_m),$$

we see that for any 1-homogeneous C^2-functions Ψ', Ψ'' satisfying

$$\max_{i,j} \|\Psi'_{x_ix_j}\|_{C^\alpha(S^{n-1})} \le 1, \quad \max_{i,j} \|\Psi''_{x_ix_j}\|_{C^\alpha(S^{n-1})} \le 1,$$

we have

$$\|P(\Psi') - P(\Psi'')\|_{L^2(S^{n-1})} \le \tag{17}$$
$$C \max_{i,j} \|\Psi'_{x_ix_j} - \Psi''_{x_ix_j}\|_{L^2(S^{n-1})} \max_{i,j} \Big(\|\Psi'_{x_ix_j}\|_{C(S^{n-1})} + \|\Psi''_{x_ix_j}\|_{C(S^{n-1})} \Big)$$

and

$$\|P(\Psi') - P(\Psi'')\|_{C^\alpha(S^{n-1})} \le \tag{18}$$
$$C \max_{i,j} \|\Psi'_{x_ix_j} - \Psi''_{x_ix_j}\|_{C^\alpha(S^{n-1})} \max_{i,j} \Big(\|\Psi'_{x_ix_j}\|_{C^\alpha(S^{n-1})} + \|\Psi''_{x_ix_j}\|_{C^\alpha(S^{n-1})} \Big).$$

This will enable us to solve the equation $Af = g$ with $g = 1 + \gamma$ by iterations if $\|\gamma\|_{C^\alpha(S^{n-1})}$ is small enough.

By $\widetilde{\Delta} f$ we shall denote the restriction of the Laplacian ΔF of F to the unit sphere. Note that the Laplacian ΔF itself is a (-1)-homogeneous function on $\mathbb{R}^n \setminus \{0\}$ (assuming again that F is twice continuously differentiable away from the origin).

Lemma 5. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every even $g = 1 + \gamma$ with $\|\gamma\|_{C^\alpha(S^{n-1})} \le \delta$, there exists $f = 1 + \varphi$ solving $Af = g$ and such that $\|\varphi\|_{C^{2+\alpha}(S^{n-1})} \le \varepsilon$ and, moreover, $\varphi = \varphi' + \varphi''$, where $\widetilde{\Delta}\varphi' = \gamma$, while $\|\varphi''\|_{L^2(S^{n-1})} \le \varepsilon \|\gamma\|_{L^2(S^{n-1})}$.*

Proof. Define the sequence φ_m as follows: $\widetilde{\Delta}\varphi_0 = \gamma$, $\widetilde{\Delta}\varphi_1 = \gamma - P(\Phi_0)$, $\widetilde{\Delta}\varphi_2 = \gamma - P(\Phi_1)$, etc., where as before, Φ_m is the 1-homogeneous extension of φ_m. Recall that by the results of Appendix I, for every even function $\chi \in C^\alpha(S^{n-1})$, there exists a unique solution ψ of the equation $\widetilde{\Delta}\psi = \chi$ and we have the estimates

$$\|\psi\|_{C^{2+\alpha}(S^{n-1})} \le K\|\chi\|_{C^\alpha(S^{n-1})}, \quad \max_{i,j} \|\Psi_{x_i x_j}\|_{L^2(S^{n-1})} \le K\|\chi\|_{L^2(S^{n-1})}$$

with some constant $K > 0$. So, all φ_m are well-defined.

Let $\delta > 0$ be a very small number. Then, under the assumption $\|\gamma\|_{C^\alpha(S^{n-1})} \le \delta$, we have

$$\|\varphi_0\|_{C^{2+\alpha}(S^{n-1})} \le K\|\gamma\|_{C^\alpha(S^{n-1})} \le K\delta.$$

Fix $\kappa \in (0, \frac{1}{K})$. It follows from (18) that as long as

$$\|\psi'\|_{C^{2+\alpha}(S^{n-1})}, \|\psi''\|_{C^{2+\alpha}(S^{n-1})} \le \frac{\kappa}{2C},$$

we have

$$\|P(\Psi') - P(\Psi'')\|_{C^\alpha(S^{n-1})} \le \kappa\|\psi' - \psi''\|_{C^{2+\alpha}(S^{n-1})}.$$

If δ is small enough, so that $K\delta < \frac{\kappa}{2C}$, we obtain

$$\|P(\Phi_0)\|_{C^\alpha(S^{n-1})} = \|P(\Phi_0) - P(0)\|_{C^\alpha(S^{n-1})} \le \kappa K\delta.$$

Hence, from $\widetilde{\Delta}(\varphi_1 - \varphi_0) = -P(\Phi_0)$, we conclude that

$$\|\varphi_1 - \varphi_0\|_{C^{2+\alpha}(S^{n-1})} \le K(\kappa K)\delta,$$

so

$$\|\varphi_1\|_{C^{2+\alpha}(S^{n-1})} \le K(1 + \kappa K)\delta.$$

If this value is still less that $\frac{\kappa}{2C}$, we can continue and write

$$\|P(\Phi_1) - P(\Phi_0)\|_{C^\alpha(S^{n-1})} \le \kappa\|\varphi_1 - \varphi_0\|_{C^{2+\alpha}(S^{n-1})} \le (\kappa K)^2\delta.$$

Thus, from $\widetilde{\Delta}(\varphi_2 - \varphi_1) = -(P(\Phi_1) - P(\Phi_0))$, we get

$$\|\varphi_2 - \varphi_1\|_{C^{2+\alpha}(S^{n-1})} \le K(\kappa K)^2\delta,$$
$$\|\varphi_2\|_{C^{2+\alpha}(S^{n-1})} \le K(1 + \kappa K + (\kappa K)^2)\delta,$$

and so on. We can continue this chain of estimates as long as

$$K(1 + \kappa K + (\kappa K)^2 + \cdots + (\kappa K)^m)\delta < \frac{\kappa}{2C},$$

which is forever if $\frac{K\delta}{1-\kappa K} < \frac{\kappa}{2C}$.

The outcome is that

$$\|\varphi_{m+1} - \varphi_m\|_{C^{2+\alpha}(S^{n-1})} \le K(\kappa K)^{m+1}\delta,$$
$$\|P(\Phi_{m+1}) - P(\Phi_m)\|_{C^{2+\alpha}(S^{n-1})} \le (\kappa K)^{m+2}\delta.$$

It follows that the sequence φ_m converges in $C^{2+\alpha}(S^{n-1})$ to some function $\varphi \in C^{2+\alpha}(S^{n-1})$ with

$$\|\varphi\|_{C^{2+\alpha}(S^{n-1})} \le \frac{K\delta}{1 - \kappa K} < \varepsilon$$

if $\delta > 0$ is small enough. This function φ will solve the equation $\widetilde{\Delta}\varphi = \gamma - P(\Phi)$, i.e., the function $f = 1 + \varphi$ will solve $Af = g$.

We put $\varphi' = \varphi_0$ and $\varphi'' = \varphi - \varphi_0$. It remains to estimate $\|\varphi''\|_{L^2(S^{n-1})} = \|\varphi - \varphi_0\|_{L^2(S^{n-1})}$. To this end, we shall use (17) instead of (18) to obtain

$$\|P(\Phi_0)\|_{L^2(S^{n-1})} = \|P(\Phi_0) - P(0)\|_{L^2(S^{n-1})} \le$$
$$\le \kappa \max_{i,j} \|(\Phi_0)_{x_i x_j}\|_{L^2(S^{n-1})} \le \kappa K\|\gamma\|_{L^2(S^{n-1})},$$

so from the equation $\widetilde{\Delta}(\varphi_1 - \varphi_0) = -P(\Phi_0)$, we obtain

$$\|\varphi_1 - \varphi_0\|_{L^2(S^{n-1})} \le \|P(\Phi_0)\|_{L^2(S^{n-1})} \le \kappa K\|\gamma\|_{L^2(S^{n-1})}$$

and

$$\|(\Phi_1)_{x_i x_j} - (\Phi_0)_{x_i x_j}\|_{L^2(S^{n-1})} \le K\|P(\Phi_0)\|_{L^2(S^{n-1})} \le K(\kappa K)\|\gamma\|_{L^2(S^{n-1})}.$$

Then

$$\|P(\Phi_1) - P(\Phi_0)\|_{L^2(S^{n-1})} \le (\kappa K)^2\|\gamma\|_{L^2(S^{n-1})},$$

and we can continue as above to get inductively the inequalities

$$\|\varphi_{m+1} - \varphi_m\|_{L^2(S^{n-1})} \le (\kappa K)^{m+1}\|\gamma\|_{L^2(S^{n-1})},$$

$$\|(\Phi_{m+1})_{x_i x_j} - (\Phi_m)_{x_i x_j}\|_{L^2(S^{n-1})} \le K(\kappa K)^{m+1}\|\gamma\|_{L^2(S^{n-1})}$$

(that requires the estimate $\max_{i,j} \|(\Phi_m)_{x_i x_j}\|_{C(S^{n-1})} \le \frac{\kappa}{2C}$, but we have already obtained that bound even for the $C^{2+\alpha}$-norm of φ_m).

Adding these estimates up, we get

$$\|\varphi - \varphi_0\|_{L^2(S^{n-1})} \le \sum_{m=0}^{\infty} \|\varphi_{m+1} - \varphi_m\|_{L^2(S^{n-1})} \le$$

$$\sum_{m=0}^{\infty} (\kappa K)^{m+1}\|\gamma\|_{L^2(S^{n-1})} = \frac{\kappa K}{1-\kappa K}\|\gamma\|_{L^2(S^{n-1})},$$

and it remains to choose $\kappa > 0$ so that $\frac{\kappa K}{1-\kappa K} < \varepsilon$. □

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