Differential Geometry, Spring 2012.

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Assignment 10.

1. **Problem 1.**

   Let $G = \{g_{ij}\}_{i,j=1,2}$ and $B = \{b_{ij}\}_{i,j=1,2}$ be the matrices of two *quadratic forms* $\boldsymbol{G}$ and $\boldsymbol{B}$ on the plane, where G is *positive definite*, and $g_{12} = g_{21}$, $b_{12} = b_{21}$. Consider a quadratic (in λ) equation $\det(B - \lambda G) = 0$. The roots λ_1 and λ_2 of this equation are called the *eigenvalues of the pair of quadratic forms* G and B.

   a) Show that if λ_1 and λ_2 are the eigenvalues of G and B, then two systems of two equations

   $$\begin{cases} (b_{11} - \lambda_i g_{11})\xi_i^1 + (b_{12} - \lambda_i g_{12})\xi_i^2 = 0 \\ (b_{12} - \lambda_i g_{12})\xi_i^1 + (b_{22} - \lambda_i g_{22})\xi_i^2 = 0, \end{cases}$$

   $i = 1, 2$, with two unknowns $\boldsymbol{f_1} = (\xi_1^1, \xi_1^2)$ and $\boldsymbol{f_2} = (\xi_2^1, \xi_2^2)$ has a non-trivial solution.

   The directions of the vectors $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ are called the *main directions* of the pair of quadratic forms; $\boldsymbol{f_1}$ corresponds to λ_1 and $\boldsymbol{f_2}$ corresponds to λ_2.

   In b) - e) you will show that *If the eigenvalues of the pair of quadratic forms are different, then the main directions are orthogonal.* In other words, you will prove that

   $$\boldsymbol{f_1} \cdot \boldsymbol{f_2} = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\xi_1^i\xi_2^j = 0, \tag{1}$$

   provided $\lambda_1 \neq \lambda_2$. Here $\boldsymbol{f_1} = \xi_1^1\boldsymbol{e_1} + \xi_1^2\boldsymbol{e_2}$, $\boldsymbol{f_2} = \xi_2^1\boldsymbol{e_1} + \xi_2^2\boldsymbol{e_2}$, and $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ are the *basic vectors* in the plane such that $\boldsymbol{e_i} \cdot \boldsymbol{e_j} = g_{ij}$, $1 \leq i, j \leq 2$.

   b) Prove that there exists a basis $\boldsymbol{d_1}$, $\boldsymbol{d_2}$ such that $\boldsymbol{d_i} \cdot \boldsymbol{d_j} = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

   c) Show that in the new basis $\boldsymbol{d_1}$, $\boldsymbol{d_2}$ for the matrices $\bar{G} = I$ and $\bar{B}$ (of $\boldsymbol{G}$ and $\boldsymbol{B}$) one has $G = A^T\bar{G}A$ and $B = A^T\bar{B}A$, where $A = \{a_{ij}\}_{i,j=1,2}$ is such that $\boldsymbol{e_i} = \sum_{j=1}^{2} a_{ij}\boldsymbol{d_j}$, $i = 1, 2$.

   d) Use c) to show that

   $$\det(B - \lambda G) = (\det A)^2 \det(\bar{B} - \lambda I), \qquad \det A \neq 0,$$

   and deduce that the conditions

   $$\det(B - \lambda G) = 0 \qquad \det(\bar{B} - \lambda I) = 0$$

   are equivalent.

   e) Prove that the quadratic form $\bar{B}$ can be brought to a form corresponding to a diagonal matrix, say W. Prove that the eigenvectors of W are orthogonal in the usual Euclidean sense. Conclude that (1) holds.

2. **Problem 2.**

   Let $S_p : T_p(S) \to T_p(S)$ be the shape operator defined by $S_p(\boldsymbol{v}) = -D_{\boldsymbol{v}}\boldsymbol{n}(p)$.

   a) Prove that S_p is *linear*.

   b) Show that the matrix representing S_p with respect to the basis $\{\boldsymbol{r_u}, \boldsymbol{r_v}\}$ is

$$II_p I_p^{-1} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

   Hint: Write $S_p(\boldsymbol{r_u}) = a\boldsymbol{r_u} + b\boldsymbol{r_v}$ and $S_p(\boldsymbol{r_v}) = c\boldsymbol{r_u} + d\boldsymbol{r_v}$, and use the definition of L, M and N to get a system of linear equations for a, b, c and d.

   c) Deduce that

$$K = \frac{LN - M^2}{EG - F^2}.$$

3. **Problem 3.**

   Compute the second fundamental forms II_p of the following parametrized surfaces. Then calculate the matrix of the shape operator, determine H and K.

   a) the cylinder

$$\boldsymbol{r}(u, v) = (a\cos u, a\sin u, v), \qquad a > 0.$$

   b) the torus

$$\boldsymbol{r}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u), \qquad 0 < r < a.$$

   c) the helicoid

$$\boldsymbol{r}(u, v) = (u\cos v, u\sin v, bv), \qquad b > 0.$$

   d) the catenoid

$$\boldsymbol{r}(u, v) = a(\cosh u\cos v, \cosh u\sin v, u).$$