Differential Geometry, Spring 2012.

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Assignment 12.

1. **Problem 1.**

   Find the principal curvatures, the principal directions, and asymptotic directions (when they exist) for each of the surfaces in Problem 3, Exercise 10. Identify the lines of curvature and asymptotic lines.

2. **Problem 2.**

   Prove that if the asymptotic directions of a surface S are orthogonal, then S is *minimal* (the mean curvature $H = 0$). Prove the converse assuming S has no planar points.

3. **Problem 3.**

   Calculate the Christoffel symbols for the following parametrized surfaces. Then check in each case that the Codazzi equations and the first Gauss equation hold.

   a) the plane, parametrized by polar coordinates $\boldsymbol{r}(u, v) = (u \cos v, u \sin v, 0)$;

   b) a helicoid $\boldsymbol{r}(u, v) = (u \cos v, u \sin v, v)$;

   c) a cone $\boldsymbol{r}(u, v) = (u \cos v, u \sin v, c\, u)$, $c \neq 0$;

   d) a surface of revolution $\boldsymbol{r}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$ with $f'(u)^2 + g'(u)^2 = 1$.

4. **Problem 4.**

   Decide whether there is a parametrized surface $\boldsymbol{r}(u, v)$ with

   a) $E = G = 1$, $F = 0$, $L = 1 = -N$, $M = 0$;

   b) $E = G = 1$, $F = 0$, $L = e^u = N$, $M = 0$;

   c) $E = 1$, $F = 0$, $G = \cos^2 u$, $L = \cos^2 u$, $M = 0$, $N = 1$.

5. **Problem 5.**

   a) Prove that in an orthogonal parametrization ($F = 0$), the Gaussian curvature K can be computed using the following formula

   $$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

   b) Conclude that for a parametrized surface with $E = G = \lambda(u, v)$ and $F = 0$, the Gaussian curvature is given by

   $$K = -\frac{1}{2\lambda}\Delta \ln \lambda, \qquad \text{where} \qquad \Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \qquad \text{is the Laplacian of } f.$$

6. **Problem 6.** (Compare with Problem 3, Exercise 8).

   Let P be a *convex polytope* in $\mathbb{R}^3$ with *faces* F_j of area $c_j (= area(F_j))$ and *unit* normals (to faces) $\boldsymbol{n}_j$.

   a) Prove that $\sum_j c_j \boldsymbol{n}_j = \mathbf{0}$.

   b) Let $(\xi, \eta, \zeta) \in \mathbb{R}^3$ be such that $\xi^2 + \eta^2 + \zeta^2 = 1$. Prove the *Cauchy projection formula*

$$area(P|(\xi, \eta, \zeta)^{\perp}) = \frac{1}{2} \sum_j c_j |(\xi, \eta, \zeta) \cdot \boldsymbol{n}_j|,$$

   where $P|(\xi, \eta, \zeta)^{\perp}$ is the *projection* of P onto the plane

$$(\xi, \eta, \zeta)^{\perp} = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (\xi, \eta, \zeta) = 0\},$$

   i.e.,

$$P|(\xi, \eta, \zeta)^{\perp} = \{(x, y, z) \in \xi^{\perp} : (x, y, z) + \lambda(\xi, \eta, \zeta) \in P \text{ for some } \lambda \in \mathbb{R}\}.$$

   c)** Let $c_j > 0$ and let $\boldsymbol{n}_j$ be *unit vectors* in $\mathbb{R}^3$, $j = 1, 2, ..., m$, $m \geq 3$, such that the condition in a) is true. Does there exist a *convex polytope* P such that c_j are areas of its faces F_j and $\boldsymbol{n}_j$ are their normals?