Differential Geometry, Spring 2012.

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Assignment 3.

1. **Problem 1.**

   Let $\boldsymbol{\alpha} \in \mathbb{R}^3$ be a trace of a regular smooth curve $\boldsymbol{\alpha}(t)$, and let P be a point on $\boldsymbol{\alpha}$. Assume that l is the line passing through P. If Q is another point on $\boldsymbol{\alpha}$, and $h = dist(Q, l)$, $d = dist(P, Q)$, the line l is called a *tangent line* provided $h/d \to 0$ as $Q \to P$. Prove that $\boldsymbol{\alpha}(t)$ has a unique tangent line, and the tangent at P corresponding to t has a direction of the vector $\boldsymbol{\alpha'}(t)$.

2. **Problem 2.**

   a) For a *helix* $x = \cos t$, $y = \sin t$, $z = t$ at the point $(1, 0, 0)$ write the equations of the tangent line, osculating plane, normal plane, normal, binormal.

   b) What is the angle between the curves $xy = c_1$ and $x^2 - y^2 = c_2$?

   c) Prove that if the trace of the plane regular curve $\boldsymbol{\gamma}$ has the angle $\pi/2$ with the curves of the family $\varphi(x, y) = const$, $\varphi_x^2 + \varphi_y^2 \neq 0$, then $dx/\varphi_x = dy/\varphi_y$.

   d) Find the equation of the circle tangent of the second order to the parabola $y = x^2$ at $(0, 0)$.

3. **Problem 3.**

   Write the equation of the osculating plane of the curve given by the equations

   $$\varphi(x, y, z) = 0, \qquad \psi(x, y, z) = 0$$

   at the point (x_o, y_o, z_o).

4. **Problem 4.**

   a) Prove that $k\tau = -\boldsymbol{T'} \cdot \boldsymbol{B'}$.

   b) Prove that $[\boldsymbol{\alpha'}, \boldsymbol{\alpha''}, \boldsymbol{\alpha'''}] = k^2\tau$, provided $\boldsymbol{\alpha}(s)$ is a unit speed curve.

   c) Let $\boldsymbol{\alpha}(t)$ be a regular curve. Suppose that there is a point $\boldsymbol{a} \in \mathbb{R}^3$ such that $\boldsymbol{\alpha}(t) - \boldsymbol{a}$ is orthogonal to $\boldsymbol{T}(t)$ for all t. Prove that $\boldsymbol{\alpha}(t)$ lies on the sphere.

   d) Let $\boldsymbol{\alpha}(t)$ be a regular curve and let $\boldsymbol{a}$ be a point that belongs to each normal plane of $\boldsymbol{\alpha}$. Prove that $\boldsymbol{\alpha}$ is a sphere a curve.

5. **Problem** 5^* **.**

   a) Prove that if all tangent lines of a given curve are parallel to a plane, then the curve is plane.

   b) Prove that if all osculating planes of a given curve pass through a fixed point, then the curve is plane.