Differential Geometry, Spring 2012.
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Assignment 4.

1. **Problem 1.**

   a) Show that
   $$\boldsymbol{\alpha}(s) = \frac{(\sqrt{1+s^2}, 2s, \ln(s+\sqrt{1+s^2})}{\sqrt{5}}$$
   is a unit speed curve and compute its Frenet-Serret apparatus.

   b) Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $k > 0$ and $\tau > 0$, and let $\boldsymbol{\beta}(s) = \int_0^s \boldsymbol{B}(\sigma)d\sigma$. Prove that $\boldsymbol{\beta}$ is unit speed and show that the Frenet-Serret apparatus $\{\bar{k}, \bar{\tau}, \bar{\boldsymbol{T}}, \bar{\boldsymbol{N}}, \bar{\boldsymbol{B}}\}$ of $\boldsymbol{\beta}$ satisfies $\bar{k} = \tau$, $\bar{\tau} = k$, $\bar{\boldsymbol{T}} = \boldsymbol{B}$, $\bar{\boldsymbol{N}} = -\boldsymbol{N}$, and $\bar{\boldsymbol{B}} = \boldsymbol{T}$.

2. **Problem 2.**

   Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $k \neq 0$. Prove that the following are equivalent:

   a) The image of $\boldsymbol{\alpha}$ lies in a plane (more simply, $\boldsymbol{\alpha}$ is a plane curve).

   b) $\boldsymbol{B}$ is a constant vector.

   c) $\tau(s) = 0$ for all s.

3. **Problem 3.**

   a) Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $k\tau \neq 0$. Prove that the tangent to the tangent spherical image is parallel to the tangent to the binormal spherical image at corresponding points.

   b) Prove that the normal spherical image of $\boldsymbol{\alpha}$ is never constant.

4. **Problem 4.**

   $\boldsymbol{\alpha}(s)$ and $\boldsymbol{\beta}(s)$ are called *Bertrand curves* if for each s_o, the normal line to $\boldsymbol{\alpha}$ at $s = s_o$ is the same as the normal line to $\boldsymbol{\beta}(s)$ at $s = s_o$, (s need not be arc length on both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$). We say that $\boldsymbol{\beta}$ is a *Bertrand mate* for $\boldsymbol{\alpha}$ if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Bertrand curves.

   a) Show that any two circles in the plane with the same center are Bertrand curves.

   b) Let
   $$\boldsymbol{\alpha}(s) = \frac{1}{2}(\arccos s - s\sqrt{1-s^2}, 1-s^2, 0),$$
   and let
   $$\boldsymbol{\beta}(s) = \frac{1}{2}(\arccos s - s\sqrt{1-s^2} - s, 1-s^2+\sqrt{1-s^2}, 0).$$
   Show that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Bertrand curves. Are they unit speed?

   c) Prove that the distance between corresponding points of a pair of Betrand curves is constant.

   d) Prove that the angle between the tangents to two Bertrand curves at corresponding points is constant.