Differential Geometry, Spring 2012.

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Assignment 5.

1. **Problem 1.**

   a) Suppose $k\tau \neq 0$ at P. Of all the planes containing the tangent line to $\boldsymbol{\alpha}$ at P, show that $\boldsymbol{\alpha}$ lies *locally* on both sides only of the osculating plane.

   b) Let $\boldsymbol{\alpha}$ be a regular curve with $k \neq 0$ at P. Prove that the planar curve obtained by projecting $\boldsymbol{\alpha}$ into its osculating plane at P has the same curvature at P as $\boldsymbol{\alpha}$.

2. **Problem 2.**

   A closed, planar curve C is said to have *constant breadth* (*or width*) μ if the distance between parallel tangent lines to C is always μ, (C need not be a circle). Assume for the rest of the problem that the curve is C^2 and $k \neq 0$.

   a) Let's call two points with parallel tangent lines opposite. Prove that if C has constant width μ, then the chord joining opposite points is normal to the curve at both points.

   Hint: If $\boldsymbol{\beta}$ is opposite to $\boldsymbol{\alpha}$, then $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \lambda(s)\boldsymbol{T}(s) + \mu\boldsymbol{N}(s)$. First explain why the coefficient of $\boldsymbol{N}$ is μ; then show that $\lambda = 0$.

   b) Prove that the sum of the reciprocals of the curvature at opposite points is equal to μ. (Warning: If $\boldsymbol{\alpha}$ is arclength-parametrized, $\boldsymbol{\beta}$ is quite unlikely to be).

3. **Problem 3.**

   Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two regular curves defined on $[a, b]$. We say $\boldsymbol{\beta}$ is an *involute* of $\boldsymbol{\alpha}$ if, for each $t \in [a, b]$,

   (i) $\boldsymbol{\beta}(t)$ lies on the tangent line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$, and

   (ii) the tangent vectors to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ at $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$, respectively, are perpendicular.

   Reciprocally, we also refer to $\boldsymbol{\alpha}$ as an *evolute* of $\boldsymbol{\beta}$.

   a) Suppose $\boldsymbol{\alpha}$ is arclength-parametrized. Show that $\boldsymbol{\beta}$ is an *involute* of $\boldsymbol{\alpha}$ if and only if $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + (c - s)\boldsymbol{T}(s)$ for some constant c. We will normally refer to the curve $\boldsymbol{\beta}$ at $s = 0$ as *the* involute of $\boldsymbol{\alpha}$. If you were to wrap a string around the curve $\boldsymbol{\alpha}$, starting at $s = 0$, the involute is the path the end of the string follows as you unwrap it, always pulling the string taut.

   b) Show that the involute of a helix is a plane curve.

   c) Show that the involute of a *catenary* (the graph of $f(x) = d\cosh(x/d)$ for some constant $d > 0$ is a *tractrix*

$$\boldsymbol{\alpha}(\theta) = (\cos\theta + \ln\tan(\theta/2), \sin\theta), \qquad \pi/2 < \theta < \pi,$$

(or $\boldsymbol{\alpha}(t) = (t - \tanh t, \operatorname{sech} t)$, $t > 0$).

d) If $\boldsymbol{\alpha}$ is an arclength-parametrized plane curve, prove that the curve $\boldsymbol{\beta}$ given by

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \frac{1}{k(s)}\boldsymbol{N}(s)$$

is the unique evolute of $\boldsymbol{\alpha}$ lying in the plane of $\boldsymbol{\alpha}$. Prove, moreover, that this curve is regular if $k' \neq 0$.