Differential Geometry, Spring 2012.

Instructor: Dmitry Ryabogin

Assignment 6.

1. **Problem 1.**

   Let $g(s)$ be a smooth function on $(-1, 1)$ and let c, d be constants. Define the curve

   $$\boldsymbol{\alpha}(t) = (c\int_0^t \sin g(s)ds, c\int_0^t \cos g(s)ds, d\,t).$$

   Prove that the ratio of the curvature and torsion of $\boldsymbol{\alpha}$ is constant.

2. **Problem 2.**

   a) Let $\boldsymbol{\alpha}$ be an arclength-parametrized *plane* curve. We create a "parallel" curve $\boldsymbol{\beta}$ by taking $\boldsymbol{\beta} = \boldsymbol{\alpha} + \epsilon\boldsymbol{N}$ (for a fixed small positive value of ϵ). Explain the terminology and express the curvature of $\boldsymbol{\beta}$ in terms of ϵ and the curvature of $\boldsymbol{\alpha}$.

   b) Now let $\boldsymbol{\alpha}$ be an arclength-parametrized *space* curve. Show that we can obtain a "parallel" curve $\boldsymbol{\beta}$ by taking $\boldsymbol{\beta} = \boldsymbol{\alpha} + \epsilon(\cos\theta\boldsymbol{N} + \sin\theta\boldsymbol{B})$ for an appropriate function θ. How many such parallel curves are there?

   c) Sketch such a parallel curve for a circular helix $\boldsymbol{\alpha}$.

3. **Problem 3.**

   a)* Suppose $\boldsymbol{\alpha}$ is arclength-parametrized and $k\tau \neq 0$. Show that $\boldsymbol{\alpha}$ has a Bertrand mate if and only if there are constants r and c so that $rk + c\tau = 1$.

   b)* Given $\boldsymbol{\alpha}$, prove that if there is more than one curve $\boldsymbol{\beta}$ so that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Bertrand mates, then there are infinitely many such curves and this occur if and only if $\boldsymbol{\alpha}$ is a circular helix.

   c) Suppose that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Bertrand mates. Prove that the torsion of $\boldsymbol{\alpha}$ and the torsion of $\boldsymbol{\beta}$ at corresponding points have constant product.

4. **Problem 4.**

   A convex plane curve with origin in its interior can be determined by its tangent lines $x\cos\theta + y\sin\theta = h(\theta)$, called *support lines*. The function $h(\theta)$ is called the support function, (here θ is in the polar coordinate, and we assume $h(\theta) > 0$ for all $\theta \in [0, 2\pi]$).

   a) Compute $h(\theta)$ for an ellipse $\{(x, y) : \frac{x^2}{16} + \frac{y^2}{9} = 1\}$ and for the l^p-ball $\{(x, y) : x^p + y^p = 1\}$, $1 \leq p \leq \infty$.

   b) Prove that the length of $\boldsymbol{\alpha}$ is given by $L = \int_0^{2\pi} h(\theta)d\theta$.

c) Prove that if C is a simple closed (convex) plane curve of constant breadth μ, then $length(C) = \pi\mu$.

d)* Prove that the line given above is tangent to the curve at the point

$$\boldsymbol{\alpha}(\theta) = (h(\theta)\cos\theta - h'(\theta)\sin\theta, h(\theta)\sin\theta + h'(\theta)\cos\theta).$$

e)* Prove that the curvature of the curve at $\boldsymbol{\alpha}(\theta)$ is $1/(h'(\theta) + h''(\theta))$.

f)* Prove that the area enclosed by $\boldsymbol{\alpha}$ is given by $A = \frac{1}{2}\int\limits_0^{2\pi}\left(h(\theta)^2 - h'(\theta)^2\right)d\theta$.

g)** Prove the isoperimetric inequality $L^2 \geq 4\pi A$.