Differential Geometry, Spring 2012.

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Assignment 7.

1. **Problem 1.**

   a) Let C be a curve and let T be the tangent line at a point $p \in C$. Draw a line parallel to the normal line at p and at a distance d of p. Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

   $$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

   where $k(p)$ is the curvature of C at p.

   Hint: Use the Taylor expansion to obtain $x(s) = s + R_x$, $y(s) = \pm ks^2/2 + R_y$.

   b) Use the previous exercise to show that if a closed plane curve C is contained inside a disk of radius r, then there exists a point $p \in C$ such that the curvature k of C at p satisfied $|k| \geq 1/r$.

   Hint: Shrink the disc till it touches the curve.

2. **Problem 2.**

   Let $\boldsymbol{\alpha}(s)$, $s \in [0, l]$ be a closed convex plane curve positively oriented, and let the curve $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \lambda \boldsymbol{N}(s)$ be parallel to $\boldsymbol{\alpha}$. Show that

   a) $l(\boldsymbol{\beta}) = l(\boldsymbol{\alpha}) + 2\pi\lambda$.

   b) $A(\boldsymbol{\beta}) = A(\boldsymbol{\alpha}) + l\lambda + \pi\lambda^2$, (here $A(\cdot)$ detones the area bounded by the corresponding curve).

3. **Problem 3.**

   Let

   $$\mathcal{L} := \{(p, \theta) \in \mathbb{R}^2 : (p\theta) \sim (p, \theta + 2k\pi) \sim (-p, \theta \pm \pi)\}.$$

   The purpose if this exercise is to show that, up to a choice of units, there is only one reasonable measure in this set.

   a) Let the *rigid motion* $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(\tilde{x}, \tilde{y}) \to (x, y)$, where

   $$x = a + \tilde{x}\cos\phi - \tilde{y}\sin\phi, \qquad y = b + \tilde{x}\sin\phi + \tilde{y}\cos\phi.$$

   Prove that F maps the line $x\cos\theta + y\sin\theta = h$ into the line

   $$\tilde{x}\cos(\theta - \phi) + \tilde{y}\sin(\theta - \phi) = h - a\cos\theta - b\sin\theta.$$

   b) Prove that F on $\mathcal{L}$ is

   $$\tilde{p} = p - a\cos\theta - b\sin\theta, \qquad \tilde{\theta} = \theta - \phi,$$

check that the Jacobian of this transformation is 1, and prove that this transformation is transitive on the set of lines in the plane, (that is, given any two lines in the plane there exists a rigid motion taking one line into the other).

c)* We define the measure of a set $\mathcal{S} \subset \mathcal{L}$ as

$$\iint\limits_{\mathcal{S}} dp\, d\theta.$$

Prove that this is, up to a constant factor, the only measure on $\mathcal{L}$ that is invariant under rigid motions.