Differential Geometry, Spring 2012.

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Assignment 8.

1. **Problem 1.**

   a) Consider a curve $\boldsymbol{\alpha}(u) = (\rho(u), z(u))$ in the (ρ, z) plane, $\rho = \rho(u) > 0$. If this curve is rotated about the z-axis, we obtain a *surface of revolution*. We may parametrize this surface as follows

   $$\boldsymbol{r}(u, v) = (\rho(u)\cos v, \rho(u)\sin v, z(u)), \qquad -\pi < v < \pi.$$

   Prove that $\boldsymbol{r}$ is a *simple surface* if the original curve $\boldsymbol{\alpha}$ was regular and one-to-one by computing $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, and $\boldsymbol{n}$.

   b) Consider the *Monge patch* $\boldsymbol{r}(u, v) = (u, v, uv)$. Find the equation of the tangent plane at the point $\boldsymbol{r}(1, 2)$. Prove that this tangent plane intersects the surface in two lines:

   $$\boldsymbol{\alpha}(t) = \boldsymbol{r}(t, 2) = (t, 2, 2t) \qquad \text{and} \qquad \boldsymbol{\beta}(t) = \boldsymbol{r}(1, t) = (1, t, t).$$

   Hint: Prove that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ do actually lie in the tangent plane.

   c) Write out the parametrization $\boldsymbol{r}(u, v)$ of the *Möbius band*. Compute $\boldsymbol{n}(u, 0)$ and show that

   $$\lim_{u \to -\pi} \boldsymbol{n}(u, 0) = -\lim_{u \to \pi} \boldsymbol{n}(u, 0)$$

   while

   $$\lim_{u \to -\pi} \boldsymbol{r}(u, 0) = \lim_{u \to \pi} \boldsymbol{r}(u, 0).$$

2. **Problem 2.**

   a) Let $S^2 = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$ and $\mathbb{R}^2 = \{(u, v, w) \in \mathbb{R}^3 : w = 0\}$. If $(u, v, 0)$ belongs to $\mathbb{R}^2$, the line determined by $(u, v, 0)$ and $(0, 0, 1)$ intersects S^2 in a point other than $(0, 0, 1)$. Denote this point by $\boldsymbol{r}(u, v)$. Compute the actual form of $\boldsymbol{r}(u, v)$ and show that $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$ is a simple surface. The inverse mapping to $\boldsymbol{r}$ is called the *stereographic projection*.

   b) Let $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^3$ be a unit speed curve with $k \neq 0$ and let

   $$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : a < u < b,\ v \neq 0\}.$$

   Define $\boldsymbol{r} : \mathcal{U} \to \mathbb{R}^3$ by $\boldsymbol{r}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\alpha}'(u)$. Prove that $\boldsymbol{r}$ is a simple surface, provided $\boldsymbol{r}$ one-to-one. It is called the *tangent developable surface* of $\boldsymbol{\alpha}$.

   c) Let

   $$\boldsymbol{r}(u, v) = (\sin u \cos v, 2\sin u \sin v, 3\cos u), \qquad -1 < u < 1,\ \ 0 < v < \pi.$$

   Show that $\boldsymbol{r}$ is a simple surface. What is it?

3. **Problem 3.**

Let P be a *convex polygon* in $\mathbb{R}^2$ with *faces* (that are actually *edges*) F_j of length $c_j = l(F_j)$ and *unit* normals (to faces) $\boldsymbol{n}_j$.

a) Prove that $\sum\limits_j c_j \boldsymbol{n}_j = \boldsymbol{0}$.

b) Let $(\xi, \eta) \in \mathbb{R}^2$ be such that $\xi^2 + \eta^2 = 1$. Prove the *Cauchy projection formula*

$$length(P|(\xi,\eta)^\perp) = \frac{1}{2}\sum_j c_j |(\xi,\eta)\cdot \boldsymbol{n}_j|,$$

where $P|(\xi,\eta)^\perp$ is the *projection* of P onto the line

$$(\xi,\eta)^\perp = \{(x,y) \in \mathbb{R}^2 : \ (x,y)\cdot(\xi,\eta) = 0\},$$

i.e.,

$$P|(\xi,\eta)^\perp = \{(x,y) \in \xi^\perp : \ (x,y) + \lambda(\xi,\eta) \in P \ \text{ for some } \ \lambda \in \mathbb{R}\}.$$

c)* Let $c_j > 0$ and let $\boldsymbol{n}_j$ be *unit vectors* in $\mathbb{R}^2$, $j = 1, 2, ..., m$, $m \geq 3$, such that the condition in a) is true. Does there exist a *convex polygon* P such that c_j are lengths of its faces F_j and $\boldsymbol{n}_j$ are their normals?