Differential Geometry, Spring 2012.

Instructor: Dmitry Ryabogin

Assignment 9.

1. **Problem 1.**

   a) Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z axis.

   b) * Show that the equation of the tangent plane at (x_o, y_o, z_o) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

   $$f_x(x_o, y_o, z_o)(x - x_o) + f_y(x_o, y_o, z_o)(y - y_o) + f_z(x_o, y_o, z_o)(z - z_o) = 0.$$

   c) Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_o = (x_o, y_o)$, is given by

   $$z = f(x_o, y_o) + f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o).$$

   Recall the definition of the differential df of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and show that the tangent plane is the graph of the differential.

2. **Problem 2.**

   If a coordinate neighborhood of a regular surface can be parametrized in the form

   $$\boldsymbol{r}(u, v) = \boldsymbol{\alpha}(u) + \boldsymbol{\beta}(v),$$

   where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

3. **Problem 3.**

   Compute the first fundamental forms of the following parametrized surfaces where they are regular:

   a) ellipsoid
   $$\boldsymbol{r}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u).$$

   b) elliptic paraboloid
   $$\boldsymbol{r}(u, v) = (au \cos v, bu \sin v, u^2).$$

   c) hyperbolic paraboloid

   $$\boldsymbol{r}(u, v) = (au \cosh v, bu \sinh v, u^2).$$

   d) hyperboloid of two sheets

   $$\boldsymbol{r}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u).$$

4. **Problem 4.**

a) Given the parametrized surface

$$\boldsymbol{r}(u,v) = (u\cos v, u\sin v, \log\cos v + u), \qquad \frac{\pi}{2} < v < \frac{\pi}{2},$$

show that the two curves $\boldsymbol{r}(u_1, v)$, $\boldsymbol{r}(u_2, v)$ determine segments of equal lengths on all curves $\boldsymbol{r}(u, const)$.

b) Show that

$$\boldsymbol{r}(u,v) = (u\sin\gamma\cos v, u\sin\gamma\sin v, u\cos\gamma), \qquad 0 < u < \infty,\ 0 < v < 2\pi,\ \gamma = const.,$$

is a parametrization of the cone with 2γ as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve

$$\boldsymbol{r}(c\exp(v\sin\gamma\,\mathrm{cotan}\,\delta), v), \qquad c = const,\ \delta = const.,$$

intersects the generators of the cone ($v = const$) under the constant angle γ.