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On a functional equation related to convex bodies with $SU(2)$-congruent projections

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Abstract Let K and L be two convex bodies in $\mathbb{R}^5$. Assume that their orthogonal projections $K|H$ and $L|H$ onto every 4-dimensional subspace H are directly $SU(2)$-congruent, i.e., they coincide up to a $SU(2)$-rotation for some complex structure in H and a translation in H. We prove that the bodies coincide up to a translation and a reflection in the origin, provided that the set of diameters of one of the bodies is contained in a finite union of two-dimensional subspaces of $\mathbb{R}^5$. We obtain this result as a consequence of a more general statement about a functional equation on the unit sphere.

Keywords Projections of convex bodies · Spherical Funk Transform · Bodies with directly congruent projections

1 Introduction

In this paper we address the following problem (cf., for example, [2, Problem 3.2, p. 125]).

Problem 1 Let $2 \leq k \leq d-1$. Assume that K and L are convex bodies in $\mathbb{R}^d$ such that the projections $K|H$ and $L|H$ are congruent for all $H \in \mathcal{G}(d,k)$. Is K a translate of $\pm L$?

Here we say that $K|H$, the projection of K onto H, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k, H)$ in H such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(d,k)$ stands for the Grassmann manifold of all k-dimensional subspaces in $\mathbb{R}^d$.

Recently, Myroshnychenko [6] together with the author gave an affirmative answer to Problem 1 in the class of polytopes. We refer the reader to [1,3,5,7] and [8], for the history and some partial results related to Problem 1.

Our first result is

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Theorem 1 *Let K and L be two convex bodies in $\mathbb{R}^5$. Assume that for every $\xi \in S^4$ the projections $K|\xi^\perp$ and $L|\xi^\perp$ are directly $SU(2)$-congruent, i.e., for every $\xi \in S^4$ there is a rotation $\varphi_\xi \in SU(2, \xi^\perp)$ for some complex structure in $\xi^\perp$ and a vector $a_\xi \in \xi^\perp$ such that*

$$\varphi_\xi(K|\xi^\perp) + a_\xi = L|\xi^\perp. \tag{1}$$

Then $K + b = L$ or $-K + b = L$ for some $b \in \mathbb{R}^5$, provided that the set of diameters of one of the bodies is contained in a finite union of two-dimensional subspaces of $\mathbb{R}^5$.

We obtain Theorem 1 as a consequence of a more general statement about a functional equation on the unit sphere. Let

$$M(g_e) = \left\{x \in S^4 : \ g_e(x) = \max_{S^4} g_e\right\} \tag{2}$$

be the set of directions of the maxima of the even part of a continuous function g defined on S^4. We have

Theorem 2 *Let f and g be two continuous functions on S^4. Assume that for every $\xi \in S^4$ there is a rotation $\varphi_\xi \in SU(2, \xi^\perp)$ for some complex structure in $\xi^\perp$ and a vector $a_\xi \in \xi^\perp$ such that*

$$f(\varphi_\xi(x)) + a_\xi \cdot x = g(x) \qquad \forall x \in S^4 \cap \xi^\perp. \tag{3}$$

Then there exists $b \in \mathbb{R}^5$ such that $f(x)+b\cdot x = g(x)$ for all $x \in S^4$ or $f(-x)+b\cdot x = g(x)$ for all $x \in S^4$, provided that $M(g_e)$ is contained in a finite union of large 1-dimensional circles of S^4.

The paper is organized as follows. In the next section we recall some definitions and prove several auxiliary Lemmata that will be used later. We prove Theorems 2 and 1 in Sects. 3 and 4.

1.1 Notation

We denote by $S^4 = \{x \in \mathbb{R}^5 : |x| = 1\}$ the set of all unit vectors in the Euclidean space $\mathbb{R}^5$. For any unit vector $\xi \in S^4$ we let $\xi^\perp$ to be the orthogonal complement of ξ in $\mathbb{R}^5$, i.e., the set of all $x \in \mathbb{R}^5$ such that $x \cdot \xi = 0$; here $x \cdot \xi$ stands for a usual scalar product of x and ξ in $\mathbb{R}^5$. The notation for the orthogonal group $O(k)$ and the special orthogonal group $SO(k)$, $k \geq 2$, is standard; $span(a_1, a_2, \dots, a_m)$ stands for a m-dimensional subspace that is a linear span of linearly independent vectors $a_1, \dots, a_m$, $m \geq 1$. We will write f_e and f_o for the even and odd parts of the function f,

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \qquad f_o(x) = \frac{f(x) - f(-x)}{2}, \quad x \in \mathbb{R}^5.$$

2 Auxiliary definitions and results

We introduce a complex structure in $\mathbb{R}^4$ by identifying it with $\mathbb{C}^2$. We will say that two bodies A and B in $\mathbb{R}^4 = \mathbb{C}^2$ are directly $SU(2)$-congruent if there exists a vector $a \in \mathbb{R}^4$ and a $SU(2)$-rotation $\varphi_{\mathbb{R}^4}$ such that $\varphi(A) + a = B$.

Consider any 4-dimensional subspace $\xi^\perp$ of $\mathbb{R}^5$ orthogonal to $\xi \in S^4$. We say that $\varphi_\xi \in SO(4, \xi^\perp)$, meaning that there exists a choice of an orthonormal basis in $\mathbb{R}^5$ and a rotation $\Phi \in SO(5)$, with a matrix written in this basis, such that the action of Φ on $\xi^\perp$ is the rotation φ_ξ in $\xi^\perp$, and the action of Φ on $l(\xi) = (\xi^\perp)^\perp$ is trivial, i.e., $\Phi(y) = y$ for every $y \in l(\xi)$.

We say that a rotation φ_ξ is in $SU(2,\xi^\perp)$ if its matrix A_ξ with respect to a certain basis in $\xi^\perp$" $=$" $\mathbb{R}^4$" $=$" $\mathbb{C}^2$ is of the form (see [9], page 130):

$$A_\xi = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}, \qquad \varphi \in [-\pi, \pi].$$

Here the invariant subspaces of φ_ξ (for $\varphi \neq 0, \pi$) are the orthogonal complex lines (two-dimensional real subspaces of $\xi^\perp$) $l_1 = l_1(\xi)$ and $l_2 = l_2(\xi)$; the restriction $\varphi_\xi|_{l_1}$ is equivalent to a multiplication by $e^{i\varphi}$, and the restriction $\varphi_\xi|_{l_2}$ is equivalent to a multiplication by $e^{-i\varphi}$.

We identify $SU(2,\xi^\perp)$ with a subgroup of $SO(4,\xi^\perp)$ of the so-called *isoclinic* rotations, [11].

Lemma 1 *If f and g verify the conditions of Theorem 2, then $f_e = g_e$ on S^4.*

Proof Comparing the even parts of Eq. (3) we have

$$f_e(\varphi_\xi(u)) = g_e(u) \quad \text{for any} \quad \xi \in S^4 \quad \text{and any} \quad u \in S^4 \cap \xi^\perp.$$

Integrating over $S^4 \cap \xi^\perp$ and using the invariance of the Lebesgue measure under rotations, we obtain

$$\int\limits_{S^4\cap\xi^\perp} f_e(\varphi_\xi(u))d\sigma(u) = \int\limits_{S^4\cap\xi^\perp} f_e(u)d\sigma(u) = \int\limits_{S^4\cap\xi^\perp} g_e(u)d\sigma(u).$$

In other words, $Ff_e = Fg_e$ on S^4, where

$$Ff_e(\xi) = \int\limits_{S^4\cap\xi^\perp} f_e(u)d\sigma(u), \qquad \xi \in S^4,$$

is the Funk transform on S^4. Since it is injective on even functions (see [4], Corollary 2.7, p. 128), we obtain the desired result. □

From now on in this section we will assume that the functions are odd.

Lemma 2 (cf. Lemma 1 [7]). *Let $z \in S^4$ and let $S^4 \cap z^\perp = \Lambda_0 \cup \Lambda_\pi$, where*

$$\Lambda_0 = \left\{\xi \in S^4 \cap z^\perp : \ f_o(x) = g_o(x) \quad \forall x \in S^4 \cap \xi^\perp\right\},$$
$$\Lambda_\pi = \left\{\xi \in S^4 \cap z^\perp : \ -f_o(x) = g_o(x) \quad \forall x \in S^4 \cap \xi^\perp\right\}.$$

Then $f_o = g_o$ on S^4 or $f_o = -g_o$ on S^4.

Proof Observe that

$$\forall x \in S^4, \qquad S^4 = \bigcup_{\{\xi\in S^4\cap z^\perp\cap x^\perp\}} (S^4 \cap \xi^\perp). \tag{4}$$

Indeed, for any $y \in S^4$ we take $\xi \in S^4 \cap z^\perp \cap x^\perp \cap y^\perp$ to obtain that $y \in S^4 \cap \xi^\perp$.

Assume that there exists $x \in S^4$ such that $(S^4 \cap z^\perp \cap x^\perp) \subset \Lambda_0$, then, using (4), we see that $f_o = g_o$ on S^4. Similarly, if there exists $x \in S^4$ such that $(S^4 \cap z^\perp \cap x^\perp) \subset \Lambda_\pi$, then, $f_o = -g_o$ on S^4.

On the other hand, if for any $x \in S^4$ there exists two directions ξ_1 and $\xi_2 \in S^4 \cap z^\perp \cap x^\perp$, $\xi_1 \neq \pm\xi_2$, such that $\xi_1 \in \Lambda_0$ and $\xi_2 \in \Lambda_\pi$, then $f_o(x) = g_o(x) = -f_o(x) = 0$. Hence, $f_o = g_o = 0$ on S^4. □

Let $z \in S^4$. Define

$$\Xi_0 = \left\{ \xi \in S^4 \cap z^\perp : \ f_o(x) + a_\xi \cdot x = g_o(x) \qquad \forall x \in S^4 \cap \xi^\perp \right\},$$

and

$$\Xi_\pi = \left\{ \xi \in S^4 \cap z^\perp : \ -f_o(x) + a_\xi \cdot x = g_o(x) \qquad \forall x \in S^4 \cap \xi^\perp \right\}.$$

Theorem 3 (cf. Theorem 1.3 [5]). *Let f and g be two odd continuous functions on S^4 and let $z \in S^4$. Assume that $S^4 \cap z^\perp = \Xi_0 \cup \Xi_\pi$. Then there exists $b \in \mathbb{R}^5$ such that for all $u \in S^4$ we have $g_o(u) = f_o(u) + b \cdot u$, or for all $u \in S^4$ we have $g_o(u) = -f_o(u) + b \cdot u$.*

Proof Since the proof is very similar to the one of Theorem 1.3, [5], we sketch it briefly. Take $n = 5$ in Theorem 1.3 and Lemma 4.3 [5]. Repeating the argument, we obtain $S^4 \cap z^\perp = \Lambda_0 \cup \Lambda_\pi$ (except an obvious difference with the definitions of Ξ_0 and Ξ_π in this note and in [5], Lemmata 3.7 and 3.8 follow without any changes). It remains to apply the previous lemma with the sets Λ_0 and Λ_π that are defined analogously to those in Lemma 4.2, [5], and with the functions $\tilde{f}_o$ and $\tilde{g}_o$ that appear in the proof of Lemma 4.3 [5]. □

3 Proof of Theorem 2

Assume at first that the set of maxima of g_e consists of two opposite points, i.e.,

$$M(g_e) = \left\{ x \in S^4 : \ g_e(x) = \max_{S^4} g_e \right\} = \{\pm z\} \tag{5}$$

for some $z \in S^4$. Consider any $\xi \in S^4 \cap z^\perp$. We claim that

$$M_\xi(f_e) = \left\{ x \in S^4 \cap \xi^\perp : \ f_e(x) = M(f_e) \right\} = \{\pm z\}. \tag{6}$$

To show (6), observe at first that

$$\max_{S^4 \cap \xi^\perp} f_e = g_e(z). \tag{7}$$

Indeed, let $y \in S^4 \cap \xi^\perp$ be such that $f_e(y) = \max_{S^4 \cap \xi^\perp} f_e > g_e(z)$. Since the identity

$$f_e(\varphi_\xi(x)) = g_e(x) \qquad \forall x \in S^4 \cap \xi^\perp, \tag{8}$$

obtained by taking even parts of (3), is equivalent to

$$f_e(y) = g_e(\varphi_\xi^{-1}(y)) \qquad \forall y \in S^4 \cap \xi^\perp, \tag{9}$$

we see that (9) does not hold, for, $f_e(y) > g_e(z) \geq g_e(\varphi_\xi^{-1}(y))$. Hence, $\max_{S^4 \cap \xi^\perp} f_e \leq g_e(z)$. Since $f_e(\varphi_\xi^{-1}(z)) = g_e(z)$, a similar argument shows that $\max_{S^4 \cap \xi^\perp} f_e$ may not be smaller than $g_e(z)$. We have proved (7).

Next, we observe that for each $\xi \in S^3 \cap z^\perp$, the set $M_\xi(f_e)$ consists of two opposite points on S^4. Indeed, if the maximum were reached at two points $y_1, y_2 \in S^4 \cap \xi^\perp$, $y_1 \neq \pm y_2$, then, using (9), we see that g_e would reach the maximum at two different points $\varphi_\xi^{-1}(y_1)$ and $\varphi_\xi^{-1}(y_2) \neq \pm \varphi_\xi^{-1}(y_1)$. This contradicts (5).

Now we show (6). If it is $\{\pm y\}$ for some $y \neq z$, $y \in S^4 \cap \xi^\perp$, we take $\zeta \in (S^3 \cap z^\perp) \setminus (S^3 \cap y^\perp)$. Since $y \notin S^4 \cap \zeta^\perp$, Eq. (8) may not hold with $\xi = \zeta$. Thus, (6) holds, and we obtain $M(f_e) = M(g_e) = \{\pm z\}$.

Using the previous identity and (8), we see that $\varphi_\xi(z) = \pm z$ for all $\xi \in S^4 \cap z^\perp$. For, $\varphi_\xi(z)$ must be a point where the maximum of f_e is reached. Hence, we can assume that for every $\xi \in S^3 \cap z^\perp$ the angle of rotation of $\varphi_\xi \in SU(2, \xi^\perp)$ is zero or π (since the rotations φ_ξ are all *isoclinic* [11], any ray r in $\xi^\perp$ emanating from the origin is not parallel to $\varphi_\xi(r)$, unless the angle of rotation is zero or π).

Thus, we can assume that for all $\xi \in S^4 \cap z^\perp$, there exists $a_\xi \in \xi^\perp$ such that

$$f(x) + a_\xi \cdot x = g(x) \qquad \forall x \in S^4 \cap \xi^\perp, \tag{10}$$

or

$$f(-x) + a_\xi \cdot x = g(x) \qquad \forall x \in S^4 \cap \xi^\perp. \tag{11}$$

The proof of Theorem 2 in the case when $M(g_e)$ consists of a pair of opposite points on S^4 now follows from Lemma 1 and Theorem 3.

Consider the general case. Assume that $M(g_e)$ is a subset A of finitely many one-dimensional large circles of S^4, $A \subset \bigcup_{j=1}^k \mathbb{S}_j, \mathbb{S}_j = S^4 \cap \Pi_j$, where Π_j is a two-dimensional subspace of $\mathbb{R}^5$.

Let $z \in A$ and let $\xi \in S^4 \cap z^\perp$. Then, $\xi^\perp \supset \Pi_j$ if and only if $\xi \in \Pi_j^\perp$, $j = 1, \dots, k$. Consider

$$G_z = (S^4 \cap z^\perp) \setminus \left(\bigcup_{j=1}^k \left(S^4 \cap \Pi_j^\perp \right) \right).$$

For every $\xi \in G_z$, the subspace $\xi^\perp$ does not contain any Π_j, and we have $\xi^\perp \cap A = \{\pm z\}$. Then, for any $\xi \in G_z$, $M_\xi(g_e) = \{\pm z\}$. Repeating the argument of the first part of the proof, we obtain (10), (11) for any $\xi \in G_z$. Since G_z is dense in $S^4 \cap z^\perp$, we have (10) and (11) for any $\xi \in S^4 \cap z^\perp$ (for any $\xi \in S^4 \cap z^\perp$ it is enough to consider a sequence of subspaces $\{\xi_k^\perp\}_{k=1}^\infty$, $\xi_k \in G_z$, $\xi_k \to \xi$ as $k \to \infty$, for which (10) or (11) holds in the corresponding $\xi_k^\perp$, and pass to the limit as $k \to \infty$; one can use a converging subsequence of $\{a_{\xi_k}\}_{k=1}^\infty$ if necessary). It remains to apply Lemma 1 and Theorem 3.

The proof of Theorem 2 is complete.

4 Proof of Theorem 1

We denote by $h_K(x)$ the support function of a convex body $K \subset \mathbb{R}^n$. For $x \in \mathbb{R}^n$ it is defined as $h_K(x) = \sup_{y \in K} x \cdot y$, ([10], page 37), and it is a homogeneous function of degree 1. The width of a set $A \subset \mathbb{R}^n$ in the direction $x \in \mathbb{R}^n$, is defined as $\omega_A(x) = h_A(x) + h_A(-x)$. A segment $[z, y] \subset K$ is called a *diameter* of the convex body K if $|z - y| = \max_{\{\theta \in S^{n-1}\}} \omega_K(\theta)$. We also define $M(\omega_L|_{S^4})$ as in (2).

We will use the following well-known properties of the support function. For every convex body K,

$$h_{K|\xi^\perp}(x) = h_K(x) \text{ and } h_{\varphi_\xi(K|\xi^\perp)}(x) = h_{K|\xi^\perp}\left(\varphi_\xi^{-1}(x)\right), \quad \forall x \in \xi^\perp, \tag{12}$$

(see, for example [2, (0.21), (0.26), pages 17–18]); here φ_ξ^{-1} stands for the inverse of $\varphi_\xi \in SO(4, \xi^\perp)$.

Theorem 1 can be reformulated in terms of support functions as follows.

Theorem 4 *Let K and L be two convex bodies in $\mathbb{R}^5$. Assume that for every $\xi \in S^4$ there is a rotation $\varphi_\xi \in SU(2, \xi^\perp)$ for some complex structure in $\xi^\perp$ and a vector $a_\xi \in \xi^\perp$ such that*

$$h_{K|\xi^\perp}\left(\varphi_\xi^{-1}(x)\right) + a_\xi \cdot x = h_{L|\xi^\perp}(x) \qquad \forall x \in \xi^\perp. \tag{13}$$

Assume also that $M(\omega_L|_{S^4})$ is contained in finitely many 1-dimensional great circles of S^4. Then there exists $b \in \mathbb{R}^5$ such that $h_K(x) + b \cdot x = h_L(x)$ for all $x \in \mathbb{R}^5$, or $h_K(x) + b \cdot x = h_L(-x)$ for all $x \in \mathbb{R}^5$.

The proof of Theorems 4 and 1 now follows directly from Theorem 3, provided we take $f = h_K$ and $g = h_L$.

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