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ORIGINAL PAPER

On bodies floating in equilibrium in every orientation

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Abstract

Ulam's problem 19 from the Scottish Book asks: *is a solid of uniform density which floats in water in every position necessarily a sphere?* We obtain several results related to this problem.

Keywords Floating bodies · Ulam's problem · Normal curvature

Mathematics Subject Classification 52A38 · 52A20.

1 Introduction

Let the density of water be 1 and assume that a convex body $K \subset \mathbb{R}^3$ of uniform density $\mathcal{D} \in (0,1)$ is submerged into water. We say that K floats in equilibrium in the direction ξ orthogonal to the water surface if the line $\ell(\xi)$ connecting the center of mass of K and the center of mass of the submerged part is parallel to ξ. We say that K floats in equilibrium in every orientation if $\ell(\xi)$ is parallel to ξ for every ξ.

The following intriguing problem was proposed by Ulam [42, Problem 19]: *If a convex body $K \subset \mathbb{R}^3$ made of material of uniform density $\mathcal{D} \in (0,1)$ floats in equilibrium in any orientation in water, must K be spherical?*

Schneider [33] and Falconer [14] showed that this is true, provided K is centrally symmetric and $\mathcal{D} = \frac{1}{2}$. However, it has been recently proven in [32] that there are non-centrally-symmetric convex bodies of density $\mathcal{D} = \frac{1}{2}$ that float in equilibrium in every orientation.

The "two-dimensional version" of the problem is also very interesting. In this case, we consider floating logs of uniform cross-section, and seek for the ones that will float in every orientation with the axis horizontal. In other words, our cross-section K is a convex set in $\mathbb{R}^2$ and the water surface is a line that cuts off a set of the given area from K. If $\mathcal{D} = \frac{1}{2}$,

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Auerbach [1] has exhibited logs with non-circular cross-section, both convex and non-convex, whose boundaries are so-called Zindler curves [49]. More recently, Bracho, Montejano and Oliveros [7] showed that for densities $\mathcal{D}$ corresponding to perimetral densities $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{2}{5}$ the answer is affirmative, while Wegner proved that for some other values of $\mathcal{D} \neq \frac{1}{2}$ the answer is negative, [45, 46]; see also related results of Várkonyi [43, 44]. Overall, the case of general $\mathcal{D} \in (0, 1)$ is notably involved and widely open.

No results in $\mathbb{R}^3$ are known for densities $\mathcal{D} \in (0, 1)$ different from $\frac{1}{2}$ and no counterexamples have been found so far. In this paper we prove and recall several results which were used in the case of density $\frac{1}{2}$, [32], and which, we believe, would help to attack the problem for other densities. We will formulate our results in terms of the volume $\delta = \mathcal{D}\mathrm{vol}_d(K)$ of the submerged part of K (see Definition 1 in Sect. 2). We begin with

Theorem 1 *Let $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$.*

If K floats in equilibrium at the level δ in every orientation, then, for all hyperplanes H that cut off the parts of volume δ from K, the cutting sections $K \cap H$ have equal moments of inertia with respect to all $(d-2)$-dimensional planes $\Pi \subset H$ passing through the center of mass of $K \cap H$ and these moments are independent of H and Π.

Conversely, let K have a C^1-smooth boundary and let the center of mass of K coincide with the center of mass of the surface of centers, i.e., the locus of the centers of mass of all parts of volume δ that are cut off by the cutting hyperplanes H. If all cutting sections $K \cap H$ have equal moments of inertia with respect to all $(d-2)$-dimensional planes $\Pi \subset H$ passing through the center of mass of $K \cap H$ and these moments are independent of H and Π, then K floats in equilibrium at the level δ in every orientation.

This Theorem gives an affirmative answer to a question mentioned in [9, p. 20, line 14 from below]: “It seems that the floating body problem is just (V, I)". The result was also recently obtained by Florentin, Schütt, Werner and Zhang in [15, Theorem 1.1], but the case $\delta = \frac{\mathrm{vol}_d(K)}{2}$ is considered under the assumption that the *Dupin floating body* $K_{[\delta]}$ coincides with the Bárány-Larman-Schütt-Werner floating body K_δ (or the *convex floating body*, see Definitions 3 and 4 in Sect. 2) and it is a single point.

An analogous Theorem for $d = 2$ was obtained by Davidov [10] and independently by Auerbach [1], see Theorem 6 and Remark 3 at the end of Sect. 4.

Corollary 1 *Let $d \geq 3$, let a convex body K have a C^1-smooth boundary and let $\delta \in (0, \mathrm{vol}_d(K))$. Assume also that the center of mass of K coincides with the center of mass of the surface of centers. If for every hyperplane H that cuts off the part of volume δ from K every cutting section $K \cap H$ is $(d+1)$-equichordal, i.e., if there exists a constant c such that for every line $\ell \subset K \cap H$ passing through the center of mass $\mathcal{C}(K \cap H)$ and having two points of intersection $\zeta_\pm(\ell)$ with the boundary of K one has*

$$\mathrm{dist}^{d+1}(\mathcal{C}(K \cap H), \zeta_+(\ell)) + \mathrm{dist}^{d+1}(\mathcal{C}(K \cap H), \zeta_-(\ell)) = c,$$

then K floats in equilibrium in every orientation.

The converse is not true, provided $\delta = \frac{\mathrm{vol}_d(K)}{2}$, i.e., there exists a *non-centrally-symmetric* body of revolution K that floats in equilibrium in every orientation (see [32, Theorem 1]), yet not every section $K \cap H$ of this body, by the hyperplane H that cuts off half of the volume, is $(d+1)$-equichordal. Indeed, if every such $K \cap H$ were $(d+1)$-equichordal, then K would be a Euclidean ball (this was shown in [31, Theorem 1] for $\delta \in (0, \mathrm{vol}_d(K))$, provided $K_{[\delta]} = K_\delta$, but one can check that a similar result holds for $\delta = \frac{\mathrm{vol}_d(K)}{2}$). This contradicts the fact that K is not centrally-symmetric.

Problem 1 *Is it possible to construct a convex body K and find $\delta \in (0, \mathrm{vol}_d(K))$ so that $K \cap H$ is $(d+1)$-equichordal for every hyperplane H that cuts off the part of volume δ from K, but K is not an Euclidean ball?*

We refer the reader to [9, pp. 9–11], [16, Chapter 6] and references therein for the information about equichordal bodies.

We also have the following result.

Corollary 2 *Let $d \geq 2$ and let a sequence $(\delta_n)_{n=1}^{\infty}$ of positive numbers be such that the Dupin floating body $K_{[\delta_n]}$ coincides with the floating body K_{δ_n} for all $n \in \mathbb{N}$ and $\delta_n \to 0$ as $n \to \infty$. If K floats in equilibrium in every orientation for all levels δ_n, then K is a Euclidean ball.*

Together with Myroshnychenko and Saroglou, [27, Theorem 1.10], we showed that if all *central* sections of an origin-symmetric convex body $K \subset \mathbb{R}^d$, $d \geq 3$, are *isotropic*, then it must be a Euclidean ball (a central section $K \cap H$ is isotropic, provided that it has equal moments of inertia with respect to all $(d-2)$-dimensional planes $\Pi \subset H$ passing through the origin (see Definition 6 in Sect. 2) and these moments are independent of H and Π). Using this result and Theorem 1, one can also give a different proof[1] of the aforementioned result of Schneider and Falconer obtained in [14, 33] via spherical harmonics. We have

Theorem 2 *Let $d \geq 3$ and let $K \subset \mathbb{R}^d$ be an origin-symmetric convex body. If K floats in equilibrium in every orientation at the level $\delta = \frac{\mathrm{vol}_d(K)}{2}$ then K is a Euclidean ball.*

Most of the results of this paper, as well as many other results on floating bodies, follow from the classical theorems of Dupin which, we believe, were missed by the mathematical community, [13, Chapter XXIV], [48, Hydrostatics, Part I]). In Sects. 2 and 3 we formulate and prove these theorems in $\mathbb{R}^d$, $d \geq 3$ (see also [32, Appendices A and B]).

We refer the interested reader to [23, pp. 90–93], [9, pp. 19–20], [16, pp. 376–377], [34, pp. 560–563], and [17], for an exposition of known results related to Ulam's Problem 19; see also [18, 20, 21, 25, 29, 30] for related results. The *floating body problems* appear in several areas of mathematics and, among other things, are related to the Busemann-Petty problems in asymptotic geometric analysis [8], to problems in statistics [28], and to problems about polytopal approximation, [3, 4, 6, 36]. We also refer the reader to [26, 35, 37–39, 47], and references therein for other works on floating bodies.

The paper is structured as follows. In the next section we recall some well-known facts about floating bodies and formulate the Theorems of Dupin in $\mathbb{R}^d$, $d \geq 3$. We prove these theorems in Sect. 3. The proofs of Lemma 1, Theorems 1 and 2, and Corollaries 1 and 2 are given in Sect. 4.

2 Notation, basic definitions and Theorems of Dupin

2.1 Notation and basic definitions

A convex body $K \subset \mathbb{R}^d$, $d \geq 2$, is a convex compact set with a non-empty interior $\mathrm{int}K$. The boundary of K is denoted by ∂K. We say that K is strictly convex if ∂K does not contain a segment. We say that K is origin-symmetric if $K = -K$ and centrally-symmetric if there exists $p \in \mathbb{R}^d$ such that $K - p = \{q - p : q \in K\}$ is origin-symmetric. For $d \geq 2$ we

[1] See [15, Theorem 1.2] for a third proof of this statement.

denote by S^{d-1} the unit sphere in $\mathbb{R}^d$ centered at the origin. Given $\xi \in S^{d-1}$ we denote by $\xi^{\perp} = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ the subspace orthogonal to ξ, where $p \cdot \xi = p_1\xi_1 + \cdots + p_d\xi_d$ is a usual inner product in $\mathbb{R}^d$. The symbol " $+$ " stands for the usual Minkowski (vector) addition, i.e., given two sets D and E in $\mathbb{R}^d$, $D + E = \{d + e : d \in D,\ e \in E\}$. Let W_j be a j-dimensional plane in $\mathbb{R}^d$, $1 \le j \le d$. The *center of mass* of a compact convex set $K \subset W_j$ with a non-empty relative interior will be denoted by $\mathcal{C}(K)$,

$$\mathcal{C}(K) = \frac{1}{\mathrm{vol}_j(K)} \int\limits_K x dx,$$

where $\mathrm{vol}_j(K)$ is the j-dimensional volume of K in $\mathbb{R}^j$. We say that a hyperplane H is the supporting hyperplane of a convex body K if $K \cap H \neq \emptyset$, but $\mathrm{int}\, K \cap H = \emptyset$. If $K \subset \mathbb{R}^d$ is a convex body containing the origin in its interior, we will use the notation

$$\rho_K(\theta) = \max\{\lambda > 0 : \lambda\theta \in K\}.$$

for the *radial function* of K. Let $m \in \mathbb{N}$. We say that a convex body K is of class $C^m(\mathbb{R}^d)$ (or K has a C^m-smooth boundary) if for every point z on the boundary ∂K of $K \subset \mathbb{R}^d$ there exists a neighborhood U_z of z in $\mathbb{R}^d$ such that $\partial K \cap U_z$ can be written as a graph of a function having all continuous partial derivatives up to the m-th order. The *Hausdorff distance* between two convex bodies K and L is defined as

$$d(K, L) = \sup_{\{\theta \in S^{d-1}\}} |h_K(\theta) - h_L(\theta)|,$$

where h_K, h_L are the *support functions* of bodies K, L, and for any $\theta \in S^{d-1}$, $h_K(\theta) = \sup\limits_{\{y \in K\}} \theta \cdot y$.

We recall several well-known facts and definitions. Let $d \ge 3$, let $K \subset \mathbb{R}^d$ be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$ be fixed. Given a direction $\xi \in S^{d-1}$ and $t = t(\xi) \in \mathbb{R}$, we call a hyperplane

$$H(\xi) = H_t(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi = t\}, \tag{1}$$

the *cutting hyperplane* of K in the direction ξ, if it cuts out of K the given volume δ, i.e., if

$$\mathrm{vol}_d(K \cap H^-(\xi)) = \delta, \qquad H^-(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi \le t(\xi)\}, \tag{2}$$

(see Fig. 1).

Now we recall the notions of *floating in equilibrium* and the *surface of centers*, [13, 48].

Definition 1 Let $\xi \in S^{d-1}$ and let $\mathcal{C}(\xi) = \mathcal{C}_\delta(\xi)$ be the center of mass of the submerged part $K \cap H^-(\xi)$ satisfying (2). A convex body K *floats in equilibrium in the direction* $\xi \in S^{d-1}$ *at the level* δ if (2) holds and the line $l(\xi)$ connecting $\mathcal{C}(K)$ with $\mathcal{C}_\delta(\xi)$ is orthogonal to the "free water surface" $H(\xi)$, i.e., the line $l(\xi)$ is "vertical" (parallel to ξ, see Fig. 1). We say that K floats in equilibrium in every orientation at the level δ if $l(\xi)$ is parallel to ξ for every $\xi \in S^{d-1}$.

Definition 2 Let K be a convex body, let $\xi \in S^{d-1}$ and let $\mathcal{C}(\xi) = \mathcal{C}_\delta(\xi)$ be the center of mass of the submerged part $K \cap H^-(\xi)$ satisfying (2). The geometric locus $\{\mathcal{C}_\delta(\xi) : \xi \in S^{d-1}\}$ is called the *surface of centers* $\mathcal{S} = \mathcal{S}_\delta$ or the *surface of buoyancy* (see Fig. 2).

One can show, see Theorem 3 below, that the surface of centers is the boundary of a strictly convex body.

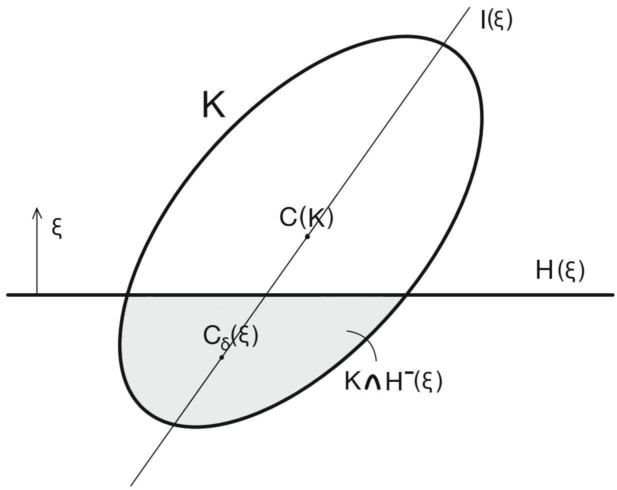

Fig. 1 A body K and its submerged part $K \cap H^-(\xi)$

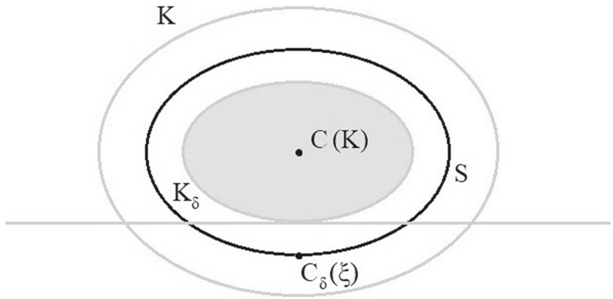

Fig. 2 Floating body K_δ and surface of centers $\mathcal{S}$

Remark 1 It was recently proved in [20] that the surface of centers $\mathcal{S}$ is C^{k+1}-smooth, provided K is of class C^k, $k \geq 0$. In particular, if K is an arbitrary convex body, then $\mathcal{S}$ is C^1-smooth.

The following result is well-known, see [17, p. 203], [43, Section 2.1] and [20, Corollary 2.4]. In the next section we give a different proof.

Lemma 1 *Let $d \geq 2$, let K be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$. If K floats in equilibrium in every orientation at the level δ, then the surface of centers $\mathcal{S}$ is a sphere. Conversely, if $\mathcal{S}$ is a sphere centered at $\mathcal{C}(K)$, then K floats in equilibrium in every orientation.*

It is known that the condition of $\mathcal{S}$ being centered at $\mathcal{C}(K)$ is satisfied for $\delta = \frac{\mathrm{vol}(K)}{2}$ ($\mathcal{C}(K)$ is an arithmetic average of $\mathcal{C}(K \cap H^+(\xi))$ and $\mathcal{C}(K \cap H^-(\xi))$ for every $\xi \in S^{d-1}$), and for any $\delta \in (0, \mathrm{vol}_d(K))$, provided K is centrally-symmetric.

Now we pass to the notion of a Dupin floating body $K_{[\delta]}$ of K. It was introduced by C. Dupin in 1822, [12].

Definition 3 A non-empty convex set $K_{[\delta]}$ is the Dupin floating body of K if each supporting plane of $K_{[\delta]}$ cuts off a set of volume $\delta \in (0, \mathrm{vol}_d(K))$ from K.

We remark that $K_{[\delta]}$ does not necessarily exist for every convex K, see [22] or [28, Chapter 5], but if K has a sufficiently smooth boundary and $\delta > 0$ is small enough, then $K_{[\delta]}$ exists, [22, Satz 2].

The notion of a *convex floating body* was introduced independently in [4, 38].

Definition 4 A body K_δ is called the convex floating body of K, provided

$$K_\delta = \bigcap_{\{\xi \in S^{d-1}\}} H^+(\xi), \qquad H^+(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi \geq t(\xi)\}.$$

If $K_{[\delta]}$ exists, then $K_{[\delta]} = K_\delta$; K_δ is allowed to be an empty set, [38]. It was proved in [26, Theorem 3, p. 334] that $K_{[\delta]} = K_\delta$ for any $0 < \delta \leq \frac{\mathrm{vol}_d(K_\delta)}{2}$, provided K is centrally-symmetric. It was also shown in [26] that the boundary of K_δ is C^2-smooth, provided the boundary of K is C^1-smooth and for every x on the boundary of K there is a unique supporting hyperplane of K through x.

Let K float in equilibrium in every orientation for some $\delta \in (0, \mathrm{vol}_d(K))$, $\delta \neq \frac{\mathrm{vol}_d(K)}{2}$. It is not clear if the additional condition $K_{[\delta]} = K_\delta$ yields an affirmative answer to Ulam's Problem 19.

2.2 Theorems of Dupin

The solution of the problem of finding the directions in which the given convex body floats in equilibrium is contained in the following three results, proved by Dupin, (cf. [48, pp. 658–660] and [10] for $d = 2$, and [13, pp. 287–288] for $d = 3$; see also [17]). For convenience of the reader, in this section we formulate these theorems for all $d \geq 3$ and include sketches of the proofs in the next section.

Let $\xi \in S^{d-1}$ and let $\mathcal{H}(\xi)$ be a tangent hyperplane to $\mathcal{S}$ at $\mathcal{C}(\xi)$ which is the center of mass of $K \cap H^-(\xi)$, see Remark 1. The First Theorem of Dupin reads as follows.

Theorem 3 *Let $d \geq 2$, $K \subset \mathbb{R}^d$ be convex, and let $\delta \in (0, \mathrm{vol}_d(K))$. If $H(\xi)$, $\xi \in S^{d-1}$, is a cutting hyperplane, then $\mathcal{H}(\xi)$ is parallel to $H(\xi)$. Moreover, the bounded set $L(\mathcal{S})$ with boundary $\mathcal{S}$ is a strictly convex body.*

The Second Theorem of Dupin is

Theorem 4 *Let $d \geq 2$, $K \subset \mathbb{R}^d$ be convex, and let $\delta \in (0, \mathrm{vol}_d(K))$. Assume that $H(\xi)$, $\xi \in S^{d-1}$, is a cutting hyperplane and $\{H_n\}_{n=1}^\infty$, $H_n = H(\xi_n)$, is any sequence of cutting hyperplanes converging to $H(\xi)$ as $\xi_n \to \xi$ for $n \to \infty$ and such that the limit $\lim\limits_{n \to \infty} H(\xi) \cap H(\xi_n)$ exists. Then the $(d-2)$-dimensional plane $\Pi = \lim\limits_{n \to \infty} H(\xi) \cap H(\xi_n)$ passes through the center of mass of $K \cap H(\xi)$.*

We remark that by writing *a sequence of cutting hyperplanes* $\{H_n\}_{n=1}^\infty$, $H_n = H_{t_n}(\xi_n)$, *converges to* $H(\xi) = H_t(\xi)$ *as* $\xi_n \to \xi$ (see (1)), we also tacitly assume that a sequence of distances to the origin $\{t_n\}_{n=1}^\infty$ converges to t when $n \to \infty$.

In order to formulate the third Theorem of Dupin in the case $d \geq 3$, we recall the notions of a *metacenter* [13, p. 284] and of a *moment of inertia* [48, p. 553].

To define the metacenter *heuristically*, assume that a body $K \subset \mathbb{R}^3$ is "cylindrical". In naval architecture, [41], a ship floating originally at a horizontal waterline $H(\xi) \subset E$ is rotated through a small angle by an external force and then floats at waterline $H(\eta) \subset E$ (it is assumed that $H(\xi)$ and $H(\eta)$ intersect at the center of mass of K). Then the point

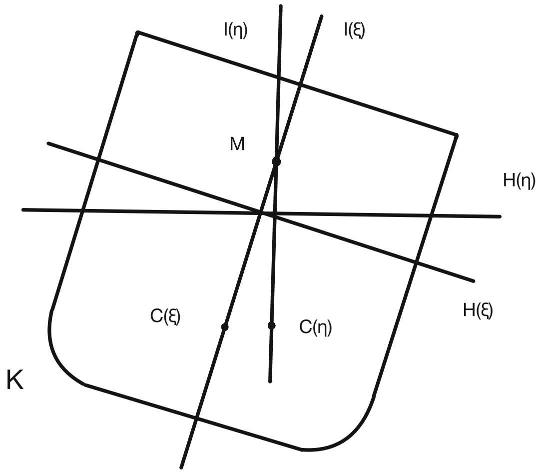

Fig. 3 The metacenter $M = l(\xi) \cap l(\eta)$ of K

$M = \ell(\xi) \cap \ell(\eta)$ is the metacenter, where $\ell(\xi)$ is the line parallel to ξ passing through the old center of boyancy $\mathcal{C}(\xi)$ and $\ell(\eta)$ is the line parallel to η passing through the new center of boyancy $\mathcal{C}(\eta)$, see Fig. 3.

Now we recall a rigorous definition, [13, pp. 284, 285].

Definition 5 Let $\mathcal{S}$ be the surface of centers and let $\mathcal{C}$ be a point on $\mathcal{S}$ at which the normal curvatures exist. Assume that $\mathcal{C}$ belongs to some curve $\gamma \subset \mathcal{S}$ with the tangent ζ at $\mathcal{C}$. Take $\mathcal{C}' \in \gamma$ close to $\mathcal{C}$ and consider the normal lines $l_{\mathcal{C}}$, $l_{\mathcal{C}'}$, to $\mathcal{S}$ at $\mathcal{C}$ and $\mathcal{C}'$. If $\mu\mu'$ is a shortest distance between these lines, $\mu \in l_{\mathcal{C}}$, $\mu' \in l_{\mathcal{C}'}$, then the limiting position of the end μ of the segment $[\mu, \mu']$, when $\mathcal{C}'$ tends to $\mathcal{C}$, is the metacenter $M_{\mathcal{C}}(\zeta)$ related to $\mathcal{C}$ in the tangential direction ζ.

Let $\mathcal{S}$ be C^2-smooth. One can assume without loss of generality that the tangent hyperplane $\mathcal{H}$ to $\mathcal{S}$ at $\mathcal{C}$ is horizontal, i.e., $\mathcal{H}$ is the $x_1 \dots x_{d-1}$-hyperplane and that $\mathcal{C}$ is the origin. Then, choosing properly the directions of the axes in $\mathcal{H}$ one can assume that the equation of $\mathcal{S}$ in a small neighborhood of $\mathcal{C}$ is

$$2x_d = k_1 x_1^2 + \cdots + k_{d-1} x_{d-1}^2 + o(x_1^2, \dots, x_{d-1}^2), \tag{3}$$

where k_j, $j = 1, \dots, d-1$, are some non-negative coefficients, $k_1 \le k_2 \le \cdots \le k_{d-1}$.

Lemma 2 *The x_d-coordinate of $M_{\mathcal{C}}(\zeta)$ is*

$$\mathcal{C}\mu = \frac{k_1\zeta_1^2 + \cdots + k_{d-1}\zeta_{d-1}^2}{k_1^2\zeta_1^2 + \cdots + k_{d-1}^2\zeta_{d-1}^2}, \qquad \textit{where} \quad \zeta = (\zeta_1, \dots, \zeta_{d-1}) \in S^{d-2}. \tag{4}$$

This formula is proved in [13, p. 285] for $d = 3$, the general case can be shown similarly. For convenience of the reader we prove (4) in Appendix.

Remark 2 We see that $\frac{1}{k_{d-1}} \le \mathcal{C}\mu \le \frac{1}{k_1}$ and that $\mathcal{C}\mu$ is equal to one of $\frac{1}{k_j}$, $j = 1, \dots, d-1$, provided ζ is one of the corresponding principal directions of $\mathcal{S}$ at $\mathcal{C}$.

We refer the reader to [34, pp. 103–106] and [40, pp. 82-89] for the definition of the principal directions and the normal curvatures. Alexandrov proved that if M is a convex body and $G(\xi)$ is its supporting hyperplane, then the normal curvatures exist at $M \cap G(\xi)$

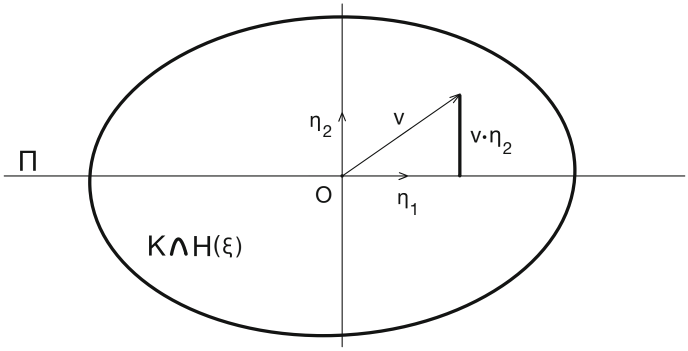

Fig. 4 Two-dimensional body $K \cap H(\xi)$ with center of mass at the origin, and a line Π parallel to η_1; we have $\mathrm{dist}(\Pi, v)^2 = |v|^2 - (v \cdot \eta_1)^2 = (v \cdot \eta_2)^2$

for almost every $\xi \in S^{d-1}$, [2, 5, 19]. Hence, for an arbitrary convex body the metacenter is defined for almost every $\xi \in S^{d-1}$.

Now we define the moment of inertia. Let $d \geq 3$, let $\delta \in (0, \frac{\mathrm{vol}_d(K)}{2})$, and let $\xi \in S^{d-1}$ be any direction. Consider a convex body K and the hyperplane $H(\xi)$ defined by (1) such that (2) holds. Choose any $(d-2)$-dimensional plane $\Pi \subset H(\xi)$ passing through the center of mass $\mathcal{C}(K \cap H(\xi))$ and let $\eta_1, \dots, \eta_{d-2}, \eta_{d-1}$ be an orthonormal basis of $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ such that

$$\Pi = \mathcal{C}(K \cap H(\xi)) + \mathrm{span}(\eta_1, \dots, \eta_{d-2}), \quad H(\xi) = \mathcal{C}(K \cap H(\xi)) + \xi^\perp. \tag{5}$$

Definition 6 The moment of inertia $I_{K\cap H(\xi)}(\Pi)$ of $K \cap H(\xi)$ with respect to Π is calculated by summing $\mathrm{dist}(\Pi, v)^2$ for every "particle" v in the set $K \cap H(\xi)$, where $\mathrm{dist}(\Pi, v) = \min\limits_{\{x\in\Pi\}} |v - x|$, (see Fig. 4), i.e.,

$$I_{K\cap H(\xi)}(\Pi) = \int\limits_{K\cap H(\xi)} \mathrm{dist}(\Pi, v)^2 dv = \int\limits_{K\cap H(\xi) - \mathcal{C}(K\cap H(\xi))} (u \cdot \eta_{d-1})^2 \, du. \tag{6}$$

The Third Theorem of Dupin reads as follows (cf. [13], p. 288).

Theorem 5 *Let $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$. If $H(\xi)$, $\xi \in S^{d-1}$, is a cutting hyperplane and $\mathcal{C} = \mathcal{C}(\xi) \in \mathcal{S}$ is the corresponding center of mass at which the normal curvatures of $\mathcal{S}$ exist in all directions and if a sequence of cutting hyperplanes $\{H_n\}_{n=1}^\infty$, $H_n = H(\xi_n)$, converging to $H(\xi)$ as $n \to \infty$, is such that the limit $\lim\limits_{n\to\infty} H(\xi) \cap H(\xi_n)$ exists, then for the corresponding sequence of the centers of mass $\{\mathcal{C}_n\}_{n=1}^\infty$, $\mathcal{C}_n = \mathcal{C}(\xi_n)$, $\mathcal{C} = \lim\limits_{n\to\infty} \mathcal{C}_n$, one has*

$$\mathcal{R}_{\mathcal{C}(\xi)}(\zeta) := dist(\mathcal{C}(\xi), M_{\mathcal{C}(\xi)}(\zeta)) = \frac{1}{\delta} I_{K\cap H(\xi)}(\Pi),$$

where $\zeta = \lim\limits_{n\to\infty} \frac{\mathcal{C}\mathcal{C}_n}{|\mathcal{C}\mathcal{C}_n|}$ and $I_{K\cap H(\xi)}(\Pi)$ is the moment of inertia of $K \cap H(\xi)$ with respect to the $(d-2)$-dimensional plane $\Pi = \lim\limits_{n\to\infty} H(\xi) \cap H(\xi_n)$.

If the reader does not want to deal with subtleties related to the almost everywhere existence of tangent hyperplanes or normal curvatures for general convex bodies, [2, 5, 19], one can assume from now on that K is C^1. In this case, $\mathcal{S}$ is C^2-smooth, [20], and Theorem 5 holds for every $\xi \in S^{d-1}$.

The following theorem can be found in [10, p. 23] and [1] in the case when K has C^1-smooth boundary. It is the Third Theorem of Dupin for $d = 2$.

Theorem 6 *Let $K \subset \mathbb{R}^2$ be convex and let $\delta \in (0, \mathrm{area}(K))$. Then*

$$R(\xi) = \frac{\mathrm{length}^3(K \cap H(\xi))}{12\,\mathrm{area}(K \cap H^-(\xi))} \quad \textit{for almost every} \quad \xi \in S^1,$$

where $H(\xi)$ and $H^-(\xi)$ are defined by (1) and (2), and $R(\xi)$ is the radius of curvature of $\mathcal{S}$ at the point of tangency $\mathcal{S} \cap \mathcal{H}(\xi)$.

3 Proofs of Theorems of Dupin

3.1 Proof of Theorem 3

Rotating and translating if necessary we can assume that ξ is such that $H(\xi)$ is "horizontal", i.e., $H(\xi) = e_d^{\perp}$. Let $\eta \in S^{d-1}$, $\eta \neq \xi$ and let $\mathcal{H}(\xi)$ be a hyperplane parallel to $H(\xi)$ and passing through $\mathcal{C}_\delta(\xi)$. We claim that $\mathcal{C}_\delta(\eta)$ is "above" $\mathcal{H}(\xi)$, i.e., $x_d(\mathcal{C}_\delta(\xi)) < x_d(\mathcal{C}_\delta(\eta))$. Since $x_d > 0\, \forall x \in (K \cap H^-(\eta)) \setminus (K \cap H^-(\xi))$ but $x_d \leq 0\, \forall x \in (K \cap H^-(\xi)) \setminus (K \cap H^-(\eta))$, we have

$$x_d(\mathcal{C}_\delta(\xi)) = \frac{1}{\delta}\Big(\int\limits_{(K\cap H^-(\xi))\setminus(K\cap H^-(\eta))} x_d dx + \int\limits_{K\cap H^-(\eta)\cap H^-(\xi)} x_d dx \Big) <$$
$$\frac{1}{\delta}\Big(\int\limits_{(K\cap H^-(\eta))\setminus(K\cap H^-(\xi))} x_d dx + \int\limits_{K\cap H^-(\eta)\cap H^-(\xi)} x_d dx \Big) = x_d(\mathcal{C}_\delta(\eta))$$

and the claim is proved. Thus, for any $\xi \in S^{d-1}$ we have $\mathcal{S} \subset \mathcal{H}^+(\xi)$, $\mathcal{S} \cap \mathcal{H}(\xi) = \mathcal{C}_\delta(\xi)$ and $\min\limits_{\{\xi \in S^{d-1}\}} |\mathcal{C}(K) - \mathcal{C}_\delta(\xi)| > 0$. We conclude that $L(\mathcal{S}) = \bigcap\limits_{\{\xi \in S^{d-1}\}} \mathcal{H}^+(\xi)$ is a strictly convex body. $\square$

3.2 Proof of Theorem 4

Rotating and translating if necessary, assume that $H(\xi)$ is "horizontal", i.e., $H(\xi) = e_d^{\perp}$. Take n large enough and consider the $(d-2)$-dimensional plane $\Pi_n = H(\xi) \cap H(\xi_n)$. Introduce the "moving" coordinates $(x_1, x_2, \dots, x_{d-1}, x_d)$ so that Π_n is the $(x_2, \dots, x_{d-1})$-plane.

Denote by $A \triangle B$ the symmetric difference of two sets A and B, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$, and let $\Lambda_n = (K \cap H(\xi)) \triangle P_{H(\xi)}(K \cap H(\xi_n))$, where $P_{H(\xi)}$ is the orthogonal projection onto $H(\xi)$. Then,

$$\begin{aligned} \varDelta V &= \mathrm{vol}_d(K \cap H^-(\xi)) - \mathrm{vol}_d(K \cap H^-(\xi_n)) \\ &= \int\limits_{K\cap H(\xi)} x_1 \tan \varepsilon_n \, dx - \int\limits_{\Lambda_n} \zeta_d \, dx = 0, \end{aligned} \tag{7}$$

where $x_1 = x_1(\xi, \xi_n)$ and $\zeta_d = \zeta_d(\xi, \xi_n)$ is an error of $x_d = x_1 \tan \varepsilon_n$ in Λ_n which is obtained during the computation of $\varDelta V$ using the first integral above (see Fig. 5; observe

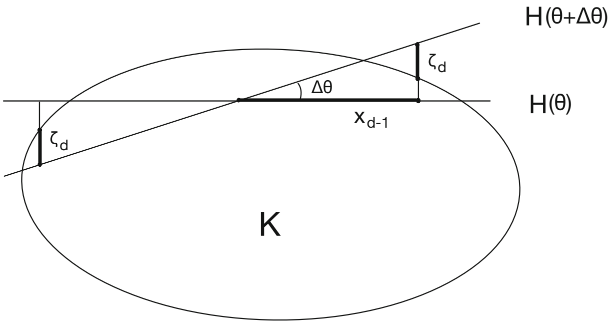

Fig. 5 The function ζ_d

that $H(\xi) \cap H(\xi_n) \cap \mathrm{int} K \neq \emptyset$ (see [30, p. 116] or [32, Appendix A])). To see (7), consider on $e_d^\perp$ an infinitesimally small element of the $(d-1)$-dimensional volume dx as a base of an infinitesimally small prism "between" $H(\xi)$ and $H(\xi_n)$ of "height" $\tan \varepsilon_n |x_1|$, where ε_n is a small angle between $H(\xi)$ and $H(\xi_n)$. The d-dimensional volume of the prism is $\tan \varepsilon_n |x_1| dx$. Summing up the volumes of the corresponding prisms we obtain (7).

By (7), we have

$$x_1(\mathcal{C}(K \cap H(\xi))) = \frac{\int\limits_{K \cap H(\xi)} x_1 \ dx}{\mathrm{vol}_{d-1}(K \cap H(\xi))} = \frac{\int\limits_{\Lambda_n} \zeta_d \ dx}{\mathrm{vol}_{d-1}(K \cap H(\xi)) \tan \varepsilon_n}.$$

Since $\mathrm{vol}_{d-1}(\Lambda_n) \to 0$ as $n \to \infty$ (see [30, p. 116] or [32, Appendix A]), and since $|\zeta_d| \leq D \tan \varepsilon_n$, where D is the diameter of K, we obtain

$$|x_1(\mathcal{C}(K \cap H(\xi)))| \leq \frac{D \tan \varepsilon_n \ \mathrm{vol}_{d-1}(\Lambda_n)}{\mathrm{vol}_{d-1}(K \cap H(\xi)) \tan \varepsilon_n} \to 0$$

as $n \to \infty$. We see that the $(d-2)$-dimensional plane $H(\xi) \cap H(\xi_n)$ tends, as $n \to \infty$, to a limiting position Π that passes through the center of mass of $K \cap H(\xi)$. □

3.3 Proof of Theorem 5

As in the previous proofs, we assume that $H(\xi) = e_d^\perp$. We take n large enough and put $\Pi_n = H(\xi) \cap H(\xi_n)$. As above we introduce the "moving" coordinates $(x_1, x_2, \dots, x_{d-1}, x_d)$ so that the $(d-2)$-dimensional plane Π_n is the $(x_2, \dots, x_{d-1})$-plane. Denote by $v_{1,n}$ and $v_{2,n}$ the d-dimensional bodies with the x_1-coordinates having opposite signs,

$$v_{1,n} = (K \cap H^-(\xi_n)) \setminus (K \cap H^-(\xi)), \quad v_{2,n} = (K \cap H^-(\xi)) \setminus (K \cap H^-(\xi_n)),$$

and let $y_{1,n}, z_{1,n}$ be the x_1-coordinates of $\mathcal{C} = \mathcal{C}_\delta(\xi)$ and $\mathcal{C}_n = \mathcal{C}_\delta(\xi_n)$, see Fig. 6(cf. Fig. 59, p. 289 from [13]). Then

$$\delta y_{1,n} = \int\limits_{K \cap H^-(\xi)} x_1 dx, \quad \delta z_{1,n} = \int\limits_{K \cap H^-(\xi_n)} x_1 dx,$$

and looking at the difference, we have

$$\delta(y_{1,n} - z_{1,n}) = \int\limits_{v_{1,n} \cup v_{2,n}} |x_1| dx.$$

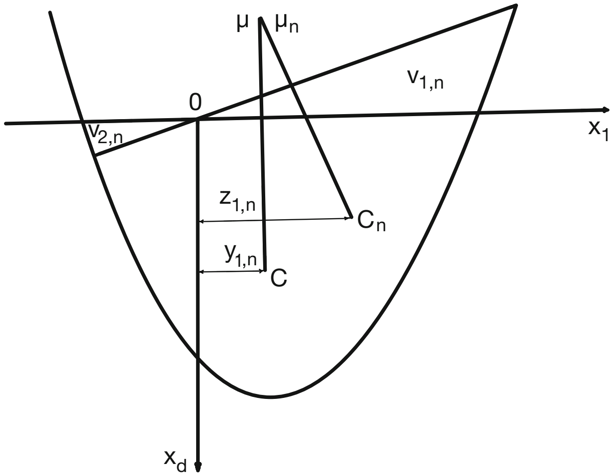

Fig. 6 The normals $\mathcal{C}\mu$ and $\mathcal{C}_n\mu_n$ to the surface of centers

Repeating the argument from the proof of Theorem 4 showing that the volumes $\mathrm{vol}_d(v_{1,n}) = \mathrm{vol}_d(v_{2,n})$ are (up to $o(\varepsilon_n)$) the sums of volumes $\varepsilon_n x_1 dx$ of infinitesimal prisms, we obtain

$$\delta(z_{1,n} - y_{1,n}) = \tan\varepsilon_n \int\limits_{K\cap H(\xi)} x_1^2 d\sigma_{d-1}(x) + o(\varepsilon_n) = \tag{8}$$

$$\tan\varepsilon_n I_{K\cap H(\xi)}(\Pi_n) + o(\varepsilon_n).$$

On the other hand, consider the normals $\mathcal{C}\mu$ and $\mathcal{C}_n\mu_n$ to $\mathcal{S}$ at the points $\mathcal{C} = \mathcal{C}_\delta(\xi)$ and $\mathcal{C}_n = \mathcal{C}_\delta(\xi_n)$. The angle ε_n between these normals is equal to the one between the hyperplanes $H(\xi)$ and $H(\xi_n)$. At the same time this is the angle between the x_d-axis and $\mathcal{C}_n\mu_n$. By definition of the metacenter, the vector $\mu\mu_n$ is "parallel" to Π_n, so μ and μ_n have the same x_1-coordinate; it is the x_1-coordinate of the intersection of orthogonal projections of lines ℓ, ℓ_n, containing $\mathcal{C}\mu$, $\mathcal{C}_n\mu_n$, onto the x_1x_d-plane. We conclude that $z_{1,n} - y_{1,n}$ is the projection of $\mathcal{C}_n\mu_n$ onto the x_1-axis, i.e., $z_{1,n} - y_{1,n} = \sin\varepsilon_n|\mathcal{C}_n\mu_n|$. Substituting this expression into (8) and passing to the limit as $n \to \infty$ we see that

$$|\mathcal{C}\mu| = \lim_{n\to\infty} |\mathcal{C}_n\mu_n| = \frac{I_{K\cap H(\xi)}(\Pi)}{\delta},$$

which is the desired conclusion. □

4 Proofs of Lemma 1, Theorems 7, 2, and Corollaries 2, 1

We start with the proof of Lemma 1 (cf. [18], [25], [17, p. 203] and [20, Corollary 2.4 and Proposition 2.2]).

Proof At first we prove the converse statement. Using the fact that all normals of the sphere intersect at its center and Theorem 3, we see that for every $\xi \in S^{d-1}$, the lines $\ell(\xi)$ passing through $\mathcal{C}(K)$ and $\mathcal{C}_\delta(\xi)$ are orthogonal to $H(\xi)$.

Now we prove the *if* part. Let $\xi \in S^{d-1}$ and let $\ell(\xi)$ be a line passing through $\mathcal{C}(K)$ and the center of mass $\mathcal{C}(\xi)$ of $K \cap H^-(\xi)$. By Theorem 3, $\mathcal{H}(\xi)$ is parallel to $H(\xi)$. Since K floats in equilibrium in the direction ξ the line $\ell(\xi)$ is orthogonal to $H(\xi)$. Since $\mathcal{H}(\xi)$ is parallel to $H(\xi)$, $\ell(\xi)$ is the normal line to $\mathcal{S}$ at $\mathcal{C}(\xi)$, and since the body floats in equilibrium in every orientation, we know that the lines $\ell(\xi)$ passing through $\mathcal{C}(K)$ are the normal lines to $\mathcal{S}$ for every $\xi \in S^{d-1}$; we recall that $\mathcal{S}$ is C^1-smooth, [20]. Consider any two-dimensional plane Π passing through $\mathcal{C}(K)$. Parametrizing the plane curve $\mathcal{S} \cap \Pi$ by the radius vector $\mathbf{r}$ going from $\mathcal{C}(K)$ to the corresponding $\mathcal{S} \cap l(\xi)$, we see that $\mathbf{r}$ is orthogonal to $\mathbf{r}'$, i.e., $\mathbf{r} \cdot \mathbf{r}' = 0$, $|\mathbf{r}|$ is constant, and $\mathcal{S} \cap \Pi$ is a circle. Since Π was chosen arbitrarily, applying [16, Corollary 7.1.4, p. 272] to $L(\mathcal{S})$ from Theorem 3, we obtain that $\mathcal{S}$ is a sphere. This gives the desired conclusion.

4.1 Proof of Corollary 2

Without loss of generality we can assume that $\mathcal{C}(K)$ is the origin. Let $\delta_n \to 0$ as $n \to \infty$ and let $\mathcal{S}_n$ be the corresponding sequence of the surfaces of centers. By the assumption, they are all spheres, say, of the radii r_n. Passing to a subsequence if necessary, we can assume that $\{r_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} r_n = r$. If $d(K_{\delta_n}, K) \to 0$ as $n \to \infty$, then, since $K_{\delta_n} \subset B^2_{r_n}(0) \subset K$, we have $d(B^2_{r_n}(0), K) \to 0$ as $n \to \infty$. Hence, K is the Euclidean ball $B^2_r(0)$.

It remains to show that $d(K_{\delta_n}, K) \to 0$ as $n \to \infty$. Let $\xi \in S^{d-1}$ and let $H_n(\xi)$ be the cutting hyperplane such that $\mathrm{vol}_d(K \cap H_n^-(\xi)) = \delta_n$. Denote by $G(\xi)$ the hyperplane that is supporting to K, orthogonal to ξ and lying in the same half-space as $K \cap H_n^-(\xi)$. We have to show that $h_n \to 0$ as $n \to \infty$, where $h_n = \max\limits_{\{\xi \in S^{d-1}\}} h_n(\xi)$, $h_n(\xi) = \mathrm{dist}(G(\xi), H_n(\xi))$.

To this end, let $\mathfrak{r} > 0$ be such that $B^2_{\mathfrak{r}}(0) \subset \mathrm{int} K$. Fix any $\xi \in S^{d-1}$ and take any $x \in K \cap G(\xi)$. Let $\mathrm{conv}(x, B^2_{\mathfrak{r}}(0))$ be a convex hull of x and $B^2_{\mathfrak{r}}(0)$. By convexity, $\mathrm{conv}(x, B^2_{\mathfrak{r}}(0)) \subset K$. Consider a cone $\Delta_n(\xi) \subset K \cap H_n^-(\xi)$ with base $\mathrm{conv}(x, B^2_{\mathfrak{r}}(0)) \cap H_n(\xi)$ and height $h_n(\xi)$. Let l_x be the line passing through the origin and x and let $y_n = l_x \cap H_n(\xi)$. Denote by $\mathcal{B}_{\tau_n}(y_n)$ the maximal $(d-1)$-dimensional ball (of radius $\tau_n = \tau_n(\xi) > 0$ and centered at y_n) contained in $\mathrm{conv}(x, B^2_{\mathfrak{r}}(0)) \cap H_n(\xi)$. We have

$$c_d \tau_n(\xi)^{d-1} h_n(\xi) \le \mathrm{vol}_d(\Delta_n(\xi)) \le \mathrm{vol}_d(K \cap H_n^-(\xi)) = \delta_n.$$

On the other hand, using similarity of triangles we see that

$$h_n(\xi) \le |x - y_n| = c(\mathfrak{r})\tau_n(\xi)|x|,$$

where $c(\mathfrak{r})$ is an absolute constant depending on $\mathfrak{r}$. These inequalities yield

$$c_d h_n^d(\xi) \le c(\mathfrak{r})^{d-1}|x|^{d-1}\delta_n \le c(\mathfrak{r})^{d-1}\mathrm{diam}(K)^{d-1}\delta_n$$

Taking the maximum of the left-hand side over $\xi \in S^{d-1}$ and passing to the limit as $n \to \infty$, we see that $h_n \to 0$ as we wanted to show. □

4.2 Proof of Theorem 1

Theorem 1 is a consequence of Lemma 1 and the Theorems of Dupin. It will be convenient to reformulate Theorem 1 in terms of the radial function.

Given a direction $\xi \in S^{d-1}$ and a hyperplane $H(\xi)$, as in (1), for which (2) holds, we will use the notation $\mathcal{P}_{K\cap H(\xi)}(w)$ for the radial function of the $(d-1)$-dimensional convex body

$K \cap H(\xi)$ with respect to the center of mass $\mathcal{C}(K \cap H(\xi))$ in the direction $w \in S^{d-1} \cap \xi^{\perp}$, i.e.,

$$\mathcal{P}_{K\cap H(\xi)}(w) = \max\{\lambda > 0 : \mathcal{C}(K \cap H(\xi)) + \lambda w \in (K \cap H(\xi))\}.$$

Theorem 7 *Let $d \geq 3$, let K be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$. If K floats in equilibrium at the level δ in every orientation, then $\forall \xi \in S^{d-1}$ the cutting sections $K \cap H(\xi)$ have equal principal moments, i.e., we have*

$$\int\limits_{S^{d-1}\cap\xi^{\perp}} w_k^2\, \mathcal{P}_{K\cap H(\xi)}^{d+1}(w)dw = (d+1)\delta\mathcal{R}, \quad k = 1, 2, \ldots, d-1, \tag{9}$$

$$\int\limits_{S^{d-1}\cap\xi^{\perp}} w_j w_k\, \mathcal{P}_{K\cap H(\xi)}^{d+1}(w)dw = 0, \quad 1 \leq k, j \leq d-1, \quad j \neq k, \tag{10}$$

where $\mathcal{R}$ is the radius of the spherical surface of centers $\mathcal{S}$.

Conversely, if K has C^1-smooth boundary, $\mathcal{C}(\mathcal{S}) = \mathcal{C}(K)$, and for every cutting hyperplane $H(\xi)$, $\xi \in S^{d-1}$, the cutting section $K \cap H(\xi)$ satisfies (2), (9) and (10) with some constant $\mathcal{R}$, then the body K floats in equilibrium in every orientation at the level δ.

Proof Let $d \geq 3$. Fix any $\xi \in S^{d-1}$ and a cutting hyperplane $H(\xi)$. Let $\Pi \subset H(\xi)$ be a $(d-2)$-dimensional plane passing through $\mathcal{C}(K \cap H(\xi))$, let $\Pi_n \subset H(\xi)$ be a sequence of $(d-2)$-dimensional planes converging and parallel to Π as $n \to \infty$, and let $H_n = H(\xi_n)$, $H_n \cap H(\xi) = \Pi_n$, be the corresponding cutting hyperplanes. If $\mathcal{C}_n = \mathcal{C}(\xi_n)$ are the centers of mass of $K \cap H_n^-$ converging to $\mathcal{C} = \mathcal{C}(\xi)$ as $n \to \infty$, then, by Theorem 5, for $\zeta = \lim\limits_{n\to\infty} \frac{\mathcal{C}\mathcal{C}_n}{|\mathcal{C}\mathcal{C}_n|}$ we have

$$\mathrm{dist}(\mathcal{C}(\xi), M_{\mathcal{C}(\xi)}(\zeta)) \overset{\text{for a.e } \xi}{=} \frac{1}{\delta} I_{K\cap H(\xi)}(\Pi). \tag{11}$$

By Lemma 1 the surface of centers $\mathcal{S}$ is a sphere of certain radius $\mathcal{R}$ centered at $\mathcal{C}(K)$. Since the radii of the normal curvatures of the sphere of radius $\mathcal{R}$ are equal to $\mathcal{R}$ at all points $\mathcal{C} \in \mathcal{S}$ in all directions and since Π was chosen arbitrarily, by Remark 2, we see that the function in the right-hand side of (11) is constant for almost every $\xi \in S^{d-1}$ and for all Π. Since the function $(\xi, \Pi) \to I_{K\cap H(\xi)}(\Pi)$ is continuous, the right-hand side of (11) is constant for every $\xi \in S^{d-1}$ and for all Π.

Hence, using (6) we obtain that for all $\xi \in S^{d-2}$ one has

$$\frac{1}{\delta} \int\limits_{K\cap H(\xi)-\mathcal{C}(K\cap H(\xi))} (v \cdot \eta_{d-1})^2\, dv = \mathcal{R} \qquad \forall \eta_{d-1} \in S^{d-1} \cap \xi^{\perp}, \tag{12}$$

where we recall that $\eta_1, \ldots, \eta_{d-2}, \eta_{d-1}$ is the orthonormal basis of $\xi^{\perp}$ such that (5) holds. Passing to polar coordinates in $H(\xi)$ with respect to $\mathcal{C}(K \cap H(\xi))$, we have

$$\int\limits_{K\cap H(\xi)-\mathcal{C}(K\cap H(\xi))} (v \cdot \eta_{d-1})^2 dv = \int\limits_{S^{d-1}\cap\xi^{\perp}} dw \int\limits_0^{\rho_{K\cap H(\xi)}(w)} (rw \cdot \eta_{d-1})^2 r^{d-2} dr = \tag{13}$$

$$\frac{1}{d+1} \int\limits_{S^{d-1}\cap\xi^{\perp}} (w \cdot \eta_{d-1})^2 \mathcal{P}_{K\cap H(\xi)}^{d+1}(w)dw, \qquad \forall \eta_{d-1} \in S^{d-1} \cap \xi^{\perp}.$$

This identity and (12) yield

$$\int\limits_{S^{d-1}\cap\xi^\perp} (w\cdot\eta_{d-1})^2\mathcal{P}^{d+1}_{K\cap H(\xi)}(w)dw = (d+1)\,\delta\,\mathcal{R}, \tag{14}$$

where the right-hand side is independent of $\eta_{d-1} \in S^{d-1}\cap\xi^\perp$. By choosing η_{d-1} to be the standard coordinate vectors in $\xi^\perp$, we obtain (9). By taking $\eta_{d-1} = (0,\dots,\underbrace{\frac{\sqrt{2}}{2}}_{j},0,\dots,0,\underbrace{\frac{\sqrt{2}}{2}}_{k},0,\dots,0)$ for different $1\le j,k\le d-1$, $j\ne k$, and using (9) we obtain (10). Since ξ was arbitrary, the proof of the *if* part is complete.

Now we prove the converse statement. Our goal is to show that the surface of centers is a sphere.

We will show at first that for almost every $\xi\in S^{d-1}$ the points $\mathcal{C}(\xi)=\mathcal{S}\cap\mathcal{H}(\xi)$ are umbilical. Let $\xi\in S^{d-1}$ be such that the normal curvatures at the corresponding point $\mathcal{C}(\xi)\in\mathcal{S}$ exist. Assume that (9) and (10) are true. We can also assume that Π satisfies (5). Then, expanding the expression $(w\cdot\eta_{d-1})^2$ by writing w in the basis $\eta_1,\dots,\eta_{d-1}$ and using the identities (12) and (13), we see that (14) holds with some constant $\mathcal{R}$ in the right-hand side, i.e., it is independent of $\eta_{d-1}\in S^{d-1}\cap\xi^\perp$. Hence, using (6), (12) and (13), we see that the right-hand side of (11) is independent of Π and ξ.

Now let ζ be any unit principal direction in the hyperplane $\mathcal{H}(\xi)$ tangent to $\mathcal{S}$ at $\mathcal{C}(\xi)$, and let Π be a two-dimensional subspace spanned by ζ and the normal to $\mathcal{S}$ at $\mathcal{C}(\xi)$. Consider a sequence of unit directions ζ_n tangent to the two-dimensional curve $\mathcal{S}\cap\Pi$ at the corresponding points $\mathcal{C}(\xi_n)\in(\mathcal{S}\cap\Pi)$ and such that $\zeta_n\to\zeta$, $\mathcal{C}(\xi_n)\to\mathcal{C}(\xi)$, as $n\to\infty$. If $\{H(\xi_n)\}_{n=1}^\infty$ is a sequence of cutting hyperplanes $H(\xi_n)$ converging to $H(\xi)$ as $n\to\infty$ with $\mathcal{C}(\xi_n)$ being the centers of mass of $K\cap H^-(\xi_n)$, applying Theorem 5 and passing to a subsequence if necessary to ensure the existence of $\lim\limits_{n\to\infty} H(\xi)\cap H(\xi_n)$, we see that the radii of the principal normal curvatures of $\mathcal{S}$ at $\mathcal{C}(\xi)$ in the principal directions are the same and the value of the radii is independent of ξ and ζ for almost every $\xi\in S^{d-1}$ and for every principal direction ζ parallel to $\mathcal{H}(\xi)$.

Thus, for almost every $\xi\in S^{d-1}$ the points $\mathcal{C}(\xi)$ are umbilical. We claim that $\mathcal{S}$ is a sphere. Indeed, recall that by Remark 1 the surface of centers is C^2. Hence, by continuity, all the points on $\mathcal{S}$ are umbilical. Using [11, Proposition 4, p. 147] and [16, Corollary 7.1.4, p. 272] we conclude that $\mathcal{S}$ must be a $(d-1)$-dimensional sphere. An application of Lemma 1 finishes the proof.

Remark 3 In the planar case an analogous result is a consequence of Lemma 1 and Theorem 6.

4.3 Proof of Corollary 1

The condition of the corollary reads as

$$\forall\xi\in S^{d-1},\qquad \mathcal{P}^{d+1}_{K\cap H(\xi)}(w)+\mathcal{P}^{d+1}_{K\cap H(\xi)}(-w)=c\qquad \forall w\in S^{d-1}\cap\xi^\perp. \tag{15}$$

The result follows from the second part of Theorem 7 by writing ρ_K^{d+1} as the sum of even and odd parts and substituting the even part from (15) into (9) and (10). ◻

4.4 Proof of Theorem 2

We recall that a measurable function $f : S^{d-1} \to \mathbb{R}$ is isotropic if the signed measure $f dx$ is isotropic, i.e., its center of mass is at the origin and the map

$$S^{d-1} \ni y \quad \to \quad \int\limits_{S^{d-1}} (y \cdot w)^2 f(w) dw$$

is constant, [24]. The following result was obtained in [27].

Theorem 8 *Let $d \geq 3$ and let $f : S^{d-1} \to \mathbb{R}$ be an even measurable function that is bounded almost everywhere. If for almost every $\xi \in S^{d-1}$ the restriction $f|_{S^{d-1} \cap \xi^\perp}$ to $S^{d-1} \cap \xi^\perp$ is isotropic (i.e., the restriction of f to almost every equator is isotropic), then f is almost everywhere equal to a constant.*

By the origin-symmetry, the centers of mass of all cutting sections are equal to the center of mass of K. Hence, we may apply Theorem 7 to see that there exists a constant c such that all second moments of the central sections $K \cap \xi^\perp$ are equal to c for all $\xi \in S^{d-1}$. The result follows from Theorem 8 with $f = \rho_K^{d+1}$. $\square$

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Appendix: Proof of Lemma 2 from [13, p. 285]

Let $\mathcal{C}$ be a point on C^2-smooth $\mathcal{S}$ and let $\gamma \subset \mathcal{S}$ be a smooth curve passing through $\mathcal{C}$. Let $\mathcal{C}' \in \gamma$ be a point infinitesimally close to $\mathcal{C}$. Consider two normal lines $\ell_{\mathcal{C}}$ and $\ell_{\mathcal{C}'}$ to $\mathcal{S}$ at $\mathcal{C}$ and $\mathcal{C}'$ and let $\mu\mu'$ be the shortest distance between these normal lines. We can assume that the tangent hyperplane to $\mathcal{S}$ at $\mathcal{C}$ is $e_d^\perp$ and that its boundary is locally described by (3).

Now drop the terms of the orders higher than 2. We have $\frac{\partial x_d}{\partial x_j} = k_j x_j$ for $j = 1, \dots, d-1$. The normal line at $\mathcal{C}' = \mathcal{C}'(x_1, \dots, x_d)$ can be expressed in terms of the "running" coordinates $(y_1, \dots, y_d)$ by equations $y_j - x_j = -k_j x_j (y_d - x_d)$, $j = 1, \dots, d-1$. The square of the distance between $(y_1, \dots, y_{d-1})$ and $\ell_{\mathcal{C}}$ is

$$\sum_{j=1}^{d-1} y_j^2 = \sum_{j=1}^{d-1} (x_j - k_j x_j (y_d - x_d))^2.$$

The "ordinate" $y_d = \mathcal{C}\mu$ of the metacenter gives the minimum of the above expression and annihilates its derivatives (at $x_d = 0$). Hence,

$$\sum_{j=1}^{d-1} k_j x_j (x_j - k_j x_j y_d) = 0, \qquad \text{i.e.,} \qquad \mathcal{C}\mu = \frac{\sum_{j=1}^{d-1} k_j x_j^2}{\sum_{j=1}^{d-1} k_j^2 x_j^2}.$$

If τ is the unit tangent vector to γ at $\mathcal{C}$, then, identifying $e_d^\perp$ with $\mathbb{R}^{d-1}$, writing τ in spherical coordinates $\zeta = (\zeta_1, \dots, \zeta_{d-1}) \in S^{d-2}$ and putting $(\zeta_1, \dots, \zeta_{d-1}) = \frac{(x_1, \dots, x_{d-1})}{\sqrt{x_1^2 + \dots + x_{d-1}^2}}$, we obtain (4).

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