ON THE p-INDEPENDENCE BOUNDEDNESS PROPERTY OF CALDERÓN-ZYGMUND THEORY

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ABSTRACT. For $0 \le \alpha < 1$ we construct examples of even integrable functions Ω on the unit sphere $\mathbf{S}^{d-1}$ with mean value zero satisfying

$$\operatorname{es\,sup}_{\xi \in \mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\theta \cdot \xi|} \, d\theta < +\infty \,,$$

such that the L^2-bounded singular integral operator T_Ω given by convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$ is not bounded on $L^p(\mathbf{R}^d)$ when $\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\alpha}{1+\alpha}$. In particular, we construct operators T_Ω that are bounded on L^p exactly when $p = 2$.

1. INTRODUCTION

Let Ω be an even complex-valued integrable function on the sphere $\mathbf{S}^{d-1}$, with mean value zero with respect to the surface measure. We discuss the L^p boundedness properties of the Calderón-Zygmund singular integral operator

$$T_\Omega(f)(x) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)\, dy = \text{p.v.} \int_{\mathbf{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)\, dy \,, \tag{1}$$

initially defined for functions f in the Schwartz class $\mathcal{S}(\mathbf{R}^d)$. The singular integral operator T_Ω is given by convolution with the distribution p.v. $\Omega(x/|x|)|x|^{-d}$ whose Fourier transform is the homogeneous of degree zero function

$$m(\Omega)(\xi) := (\text{p.v.}\, \Omega(x/|x|)|x|^{-d})\widehat{\ }(\xi) = \int_{\mathbf{S}^{d-1}} \Omega(\theta) \log \frac{1}{|\xi \cdot \theta|} \, d\theta \,. \tag{2}$$

The L^2 boundedness of T_Ω is equivalent to the condition that

$$m(\Omega) \in L^\infty(\mathbf{R}^d) \,, \tag{3}$$

i.e. $m(\Omega)$ is an essentially bounded function. Calderón and Zygmund [1], [2] have developed the theory of such singular integrals and have established their L^p boundedness in the range $1 < p < \infty$ for Ω in $L \log L(\mathbf{S}^{d-1})$. The more difficult issue of the weak type $(1,1)$ boundedness of such singular integrals with Ω in $L \log L(\mathbf{S}^{d-1})$ was settled by Christ and Rubio de Francia [5] and Seeger [14].

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The $\Omega \in L\log L$ condition is the sharpest possible, in some sense, that implies L^p boundedness for T_Ω in the whole range of $p \in (1,\infty)$, as indicated by Weiss and Zygmund [16]. A fundamental question in the subject is whether there exist other conditions on Ω that are L^p "sensitive", i.e. they imply that T_Ω is bounded on L^p for some p but not on L^q for some other index q. This question is motivated by the well-known p-independence boundedness property of Calderón-Zygmund operators with sufficiently smooth kernels, i.e. the fact that boundedness on one L^{p_0} implies boundedness on all L^p with $1 < p < \infty$. It has been an open question whether p-independence is still valid for all homogeneous singular integrals and if it fails, what condition is sensitive enough to differentiate boundedness between different L^p spaces.

A starting point for investigating this question is to ask whether (3) or even the slightly stronger condition

$$\operatorname*{essup}_{\xi\in\mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log \frac{1}{|\xi\cdot\theta|}\, d\theta < \infty\,, \tag{4}$$

suffices to imply that T_Ω is bounded on $L^p(\mathbf{R}^d)$ for some $p \neq 2$. In this work we prove that this is not the case. In fact, we construct examples of functions Ω that satisfy (4) such that T_Ω is bounded on $L^p(\mathbf{R}^d)$ exactly when $p = 2$. This answers the above question in the negative: *The p-independence boundedness property of Calderón-Zygmund theory fails for insufficiently smooth homogeneous kernels.*

Next, for $\alpha \geq 0$ we denote by

$$m_\alpha(\Omega)(\xi) = \int_{\mathbf{S}^{d-1}} |\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\theta\cdot\xi|}\, d\theta$$

a function defined for $\xi \in \mathbf{S}^{d-1}$ that measures the integrability of Ω against a power of the logarithm that appears in (4) and we introduce the following conditions on Ω:

$$\operatorname*{essup}_{\xi\in\mathbf{S}^{d-1}} m_\alpha(\Omega)(\xi) < +\infty\,. \tag{5}$$

These conditions become stronger as α increases and are known to imply boundedness for T_Ω when $1/p$ lies in some nontrivial open interval centered at $1/2$, see [10], [8].

We sharpen the aforementioned result by showing the existence of functions Ω that satisfy (5) for some $\alpha > 0$ such that the corresponding operators T_Ω are unbounded on $L^p(\mathbf{R}^d)$ for p away from 2. Precisely, we have the following:

Theorem 1. *For every α satisfying $0 \leq \alpha < 1$ there is an even integrable function Ω on $\mathbf{S}^{d-1}$ with mean value zero that satisfies (5) such that the singular integral operator T_Ω is* unbounded *on $L^p(\mathbf{R}^d)$ whenever*

$$\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\alpha}{1+\alpha}\,. \tag{6}$$

In particular, there is a function Ω such that T_Ω is L^p bounded exactly when $p = 2$.

2. The counterexample of Theorem 1

We only discuss the proof when $d = 2$ as this example can be embedded in higher dimensional spaces. We fix an $0 \le \alpha < 1$. We construct a sequence of even functions Ω_n with mean value zero such that $m(\Omega_n)$ fails to be an L^p multiplier for any p satisfying (6) while Ω_n satisfies condition (5) uniformly in n.

To prove that the norm of a Fourier multiplier on $\mathbf{R}^2$ is large, we are going to use a deLeeuw type transference argument (see [7]) in a way similar to that used in the article of Lebedev and Olevskii [11].

Lemma 1. *Let $\{x_k\}_{k=1}^l$ be an arithmetic progression in $\mathbf{R}^2$. (This means that there is a vector v such that $x_k + v = x_{k+1}$ for $k = 1, \dots, l-1$.) Let m be a function on $\mathbf{R}^2$ which is continuous at the points x_k. We define a Fourier multiplier sequence b on $\mathbf{Z}$ by the formula $b(k) = m(x_k)$ for $1 \le k \le l$ and $b(k) = 0$ otherwise. Then for some $c_p > 0$, dependent only on $1 < p < \infty$, the $L^p(\mathbf{R}^2)$ multiplier norm $\|m\|_{\mathcal{M}_p(\mathbf{R}^2)}$ of m is at least c_p times the $L^p(\mathbf{T})$ multiplier norm $\|\{b(k)\}_{k\in\mathbf{Z}}\|_{\mathcal{M}_p(\mathbf{Z})}$ of the sequence b.*

Proof. To prove the lemma, by applying a translation, a dilation and a rotation, we may assume that the points $\{x_k\}_{k=1}^l$ are the points $\{(k,0)\}_{k=1}^l$. Let R be the rectangle $[-\frac{1}{4}, l+\frac{1}{4}] \times [-\frac{1}{4}, \frac{1}{4}]$. The continuity of m at the points $\{(k,0)\}_{k=1}^l$ allows us to use a classical transference theorem (see for instance [9] Theorem 3.6.7) to deduce that

$$c_p^{-1}\|m\|_{\mathcal{M}_p(\mathbf{R}^2)} \ge \|m\chi_R\|_{\mathcal{M}_p(\mathbf{R}^2)} \ge \|\{b(k)\}_{k\in\mathbf{Z}}\|_{\mathcal{M}_p(\mathbf{Z})}\,.$$

□

If $p \ne 2$, the Riesz basis $\{e^{2\pi i k x}\}_{k=-\infty}^{\infty}$ of $L^p(\mathbf{T})$ is not unconditional. That means that for every $n = 1, 2, \dots$ there are two sequences a_k^n and $|\varepsilon_k^n| \le 1$ supported on the set $\{1, \dots, n\}$ such that

$$\Big\| \sum_{k=1}^n \varepsilon_k^n a_k^n e^{2\pi i k x}\Big\|_{L^p[0,1]} \ge K(n)\Big\| \sum_{k=1}^n a_k^n e^{2\pi i k x}\Big\|_{L^p[0,1]}, \tag{7}$$

where $K(n) \to \infty$ as $n \to \infty$. Using properties of the Rademacher functions we see that we can take $K(n) = c_p' n^{|1/2-1/p|}$, where c_p' depends only on p.

Moreover, we can choose $(\varepsilon_k^n)_{k=1}^n$ such that

$$\|(...,0,...,0,\varepsilon_1^n,\varepsilon_2^n,...,\varepsilon_n^n,0,...)\|_{M^p(\mathbf{Z})} =$$
$$\sup\{\|(...,0,...,0,\delta_1^n,\delta_2^n,...,\delta_n^n,0,...)\|_{M^p(\mathbf{Z})} : |\delta_k^n| \le 1,\ \ k = 1,...,n\}.$$

We fix $n \ge 1000$. Denote by I the angular sector from $\pi/2$ to $3\pi/4$. Fix s_0 large enough. For $k \in \{1, 2, \dots, 2n\}$, we introduce points $x_k = (t_k, s_0) \in \mathbf{Z}^2 \cap I$ and disjoint open cones I^k centered at the origin whose bisector passes through of the points x_k so that (a) the arc $I^k \cap \mathbf{S}^1$ has length roughly $(20n)^{-1}$ and (b) the distance between the points $x_k/|x_k|$ and $x_{k+1}/|x_{k+1}|$ is about n^{-1}.

For $x \in \mathbf{R}^2$ let

$$x^\perp = \{y \in \mathbf{R}^2 :\ x \cdot y = 0\}.$$

Now let $\tilde{x}_k$ be the single element of $x_k^\perp \cap \mathbf{S}^1$ which lies in the first quadrant (its argument lies in $(0, \pi/4)$) and let $A_k^{\epsilon_n}$, $k = 1, ..., 2n$, be pairwise disjoint arcs of small length ϵ_n (to be chosen later) contained in $\mathbf{S}^1$ and centered at $\tilde{x}_k$.

Define

$$\omega_k^{\epsilon_n} = C(n,\epsilon_n)\sum_{j=0}^{3}(-1)^j\chi_{A_k^{\epsilon_n}+\frac{j\pi}{2}}\,, \tag{8}$$

where $A_k^{\epsilon_n}+\frac{j\pi}{2}$ are the translations of the arcs $A_k^{\epsilon_n}$ along $\mathbf{S}^1$ by the amounts $\frac{j\pi}{2}$. Finally we introduce the function

$$\Omega_n = \sum_{k=1}^{2n}(-1)^k\varepsilon^n_{[(k+1)/2]}\omega_k^{\epsilon_n}\,,$$

where [ ] denotes the integer part. The normalization constant $C(n,\epsilon_n)$ is chosen so that

$$m_\alpha(\omega_k^{\epsilon_n})(x_k/|x_k|) = 1/2\,.$$

Set

$$D(n,\epsilon_n) = m(\omega_k^{\epsilon_n})(x_k) = m(\omega_k^{\epsilon_n})(x_k/|x_k|)$$

and note that in view of rotational invariance the constants $C(n,\epsilon_n)$ and $D(n,\epsilon_n)$ do not depend on k. Also notice that

$$\sup_x m_\alpha(\omega_k^{\epsilon_n})(x) = m_\alpha(\omega_k^{\epsilon_n})(x_k/|x_k|) \text{ and}$$
$$\sup_x |m(\omega_k^{\epsilon_n})(x)| = |m(\omega_k^{\epsilon_n})(x_k)| = |m(\omega_k^{\epsilon_n})(x_k/|x_k|)|.$$

We will make use of the following auxiliary estimates.

Lemma 2. *We have*

$$C(n,\epsilon_n)\approx \epsilon_n^{-1}|\log\epsilon_n|^{-1-\alpha}$$

and

$$D(n,\epsilon_n)\approx|\log\epsilon_n|^{-\alpha},$$

for all $x\notin \cup_{j=0}^3(I^k+\frac{j\pi}{2})\cap\mathbf{S}^1$

$$m_\alpha(\omega_k^{\epsilon_n})(x)\lesssim(\log n)^{1+\alpha}|\log\epsilon_n|^{-1-\alpha} \tag{9}$$

and for $x\notin(\cup_{j=0}^3(I^{2k}+\frac{j\pi}{2}))\cup(\cup_{j=0}^3(I^{2k-1}+\frac{j\pi}{2}))\cap\mathbf{S}^1$ *and* $1\le k\le n$

$$|\varepsilon_k^n m(\omega_{2k}^{\epsilon_n})(x)-\varepsilon_k^n m(\omega_{2k-1}^{\epsilon_n})(x)|\lesssim\frac{|\log\epsilon_n|^{-1-\alpha}}{n|x-x_{2k}|}\,. \tag{10}$$

Proof. These estimates are straightforward. To prove (10) we use the mean value theorem for integrals and we estimate $|\log u-\log v|$ by $c\,\frac{|u-v|}{|v|}$ when $|u-v|$ is small. ☐

Recall the fixed constants ε_k^n in (7). We examine properties of the function Ω_n. Observe that

$$\|\Omega_n\|_{L^1(\mathbf{S}^1)}\lesssim n\,|\log\epsilon_n|^{-1-\alpha} \tag{11}$$

and note that in view of (9) we have

$$\|m_\alpha(\Omega_n)\|_{L^\infty(\mathbf{S}^1)}\lesssim n\,(\log n)^{1+\alpha}\,|\log\epsilon_n|^{-1-\alpha}\,. \tag{12}$$

On the other hand,

$$m(\Omega_n)(x_k) = D(n,\epsilon_n)\varepsilon_k^n + \sum_{1\le i\neq k\le 2n} (-1)^i \varepsilon^n_{[(i+1)/2]} m(\omega_i^{\epsilon_n})(x_k) = D(n,\epsilon_n)\varepsilon_k^n + o_k^n,$$

and (10) implies $|o_k^n| \le D(n,\epsilon_n)/4$ as long as

$$\text{constant}\,(\log n)\,|\log\epsilon_n|^{-1-\alpha} \le \frac{1}{4}|\log\epsilon_n|^{-\alpha}$$

which is equivalent to

$$n^{4\text{constant}} \lesssim \epsilon_n^{-1}\,. \tag{13}$$

The function $m(\Omega_n)$ is continuous at the points x_k, since $m(\omega_k^{\epsilon_n})$ is continuous at $x_k/|x_k|$ as a circular convolution of $L^1(\mathbf{S}^1)$ and $L^\infty(\mathbf{S}^1)$ functions. By Lemma 1 applied to points x_k we get that the L^p multiplier norm of $m(\Omega_n)$ is comparable to $C_pD(n,\epsilon_n)n^{|\frac{1}{2}-\frac{1}{p}|}$. Indeed,

$$\begin{aligned}
\|m(\Omega_n)\|_{M^p(\mathbf{R}^2)} &\ge c_p\|(\dots,0,m(\Omega_n)(x_1),m(\Omega_n)(x_2),\dots,m(\Omega_n)(x_n),0,\dots)\|_{M^p(\mathbf{Z})}\\
&\ge c\,D(n,\epsilon_n)\Big(\|(\dots,0,\dots,0,\varepsilon_1^n,\varepsilon_2^n,\dots,\varepsilon_n^n,0,\dots)\|_{M^p(\mathbf{Z})} -\\
&\qquad\qquad \|(\dots,0,o_1^n,o_2^n,\dots,o_n^n,0,\dots)/D(n,\epsilon_n)\|_{M^p(\mathbf{Z})}\Big)\\
&\ge \frac{1}{2}\,c\,D(n,\epsilon_n)n^{|\frac{1}{2}-\frac{1}{p}|},
\end{aligned}$$

since the inequality

$$\Big\|\frac{(\dots,0,o_1^n,o_2^n,\dots,o_n^n,0,\dots)}{D(n,\epsilon_n)}\Big\|_{M^p(\mathbf{Z})} > \frac{1}{2}\,\big\|(\dots,0,\varepsilon_1^n,\varepsilon_2^n,\dots,\varepsilon_n^n,0,\dots)\big\|_{M^p(\mathbf{Z})}$$

would contradict the choice of $(\varepsilon_k^n)_{k=1}^n$. This shows that the L^p operator norm of T_{Ω_n} is at least a constant multiple of

$$D(n,\epsilon_n)n^{|\frac{1}{2}-\frac{1}{p}|} \approx |\log\epsilon_n|^{-\alpha} n^{|\frac{1}{2}-\frac{1}{p}|}\,. \tag{14}$$

We select ϵ_n satisfying

$$n(\log n)^{(1+\alpha)} = |\log\epsilon_n|^{1+\alpha}\,,$$

and we note that (13) holds for this choice of ϵ_n. Also observe that (12) gives $\|m_\alpha(\Omega_n)\|_{L^\infty(\mathbf{S}^1)} \lesssim 1$, (11) yields $\|\Omega_n\|_{L^1(\mathbf{S}^1)} \lesssim 1$, while (14) gives

$$\|T_{\Omega_n}\|_{L^p\to L^p} \ge n^{|\frac{1}{2}-\frac{1}{p}|-\frac{\alpha}{1+\alpha}}(\log n)^{-\alpha}\,.$$

We conclude that $\|T_{\Omega_n}\|_{L^p\to L^p}$ goes to infinity with n as long as $|\frac{1}{2}-\frac{1}{p}| > \frac{\alpha}{1+\alpha}$.

The existence of the function Ω claimed in Theorem 1 is a consequence of the uniform boundedness principle. Denote by $\mathcal{B}_\alpha$ the Banach space of all even integrable functions Ω on $\mathbf{S}^1$ with mean value zero with norm

$$\|\Omega\|_{\mathcal{B}_\alpha} \equiv \|\Omega\|_{L^1(\mathbf{S}^1)} + \|m_\alpha(\Omega)\|_{L^\infty(\mathbf{S}^1)} < \infty.$$

Consider the family of linear maps from $\mathcal{B}_\alpha \to L^p(\mathbf{R}^n)$

$$\Omega \to T_\Omega(f)$$

indexed by functions in the set $U = \{f \in L^p(\mathbf{R}^n) : \|f\|_{L^p} = 1\}$. If no Ω as in Theorem 1 existed, then for all $\Omega \in \mathcal{B}_\alpha$ we would have

$$\sup_{f\in U} \|T_\Omega(f)\|_{L^p} \le C(\Omega) < \infty\,.$$

The uniform boundedness principle implies the existence of a constant $K < \infty$ such that

$$\|T_\Omega\|_{L^p\to L^p} = \sup_{f\in U} \|T_\Omega(f)\|_{L^p} \le K\,\|\Omega\|_{\mathcal{B}_\alpha}$$

for all $\Omega \in \mathcal{B}_\alpha$. But this clearly contradicts the construction of the Ω_n's whenever $|\frac{1}{2} - \frac{1}{p}| > \frac{\alpha}{1+\alpha}$. This concludes the proof of Theorem 1.

3. Final Remarks

It is natural to ask whether boundedness holds for T_Ω outside the region ruled out by Theorem 1. This question was previously addressed and partially answered by Grafakos and Stefanov [10] who showed that condition (5) implies the boundedness of T_Ω on $L^p(\mathbf{R}^d)$ for p satisfying $\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{\alpha}{2(2+\alpha)}$. A sharper version of this theorem where $\frac{\alpha}{2(2+\alpha)}$ is replaced by $\frac{\alpha}{2(1+\alpha)}$ was obtained by Fan, Guo, and Pan [8].

The issue of the sufficiency of condition (5) for the L^p boundedness of T_Ω remains unanswered for p's satisfying $\frac{\alpha}{2(1+\alpha)} \le |\frac{1}{2} - \frac{1}{p}| \le \frac{\alpha}{1+\alpha}$ whenever $0 < \alpha < 1$. It is possible that for $\alpha > 1$, T_Ω is bounded on L^p for all $1 < p < \infty$ whenever (5) is satisfied, but this is also unknown at present. We note that for zonal functions Ω, condition (4) suffices for the boundedness of T_Ω on all L^p spaces $(1 < p < \infty)$ as proved by Ryabogin and Rubin [13].

It should be noted that the counterexamples discussed in this paper are related to those that indicate the sharpness in the Coifman-Rubio de Francia-Semmes condition [6] in terms of the s-variation of the multipliers. They are also related in spirit to the work of Carbery, Christ, Vance, Wainger, and Watson [3], Christ [4], Seeger, Wainger, Wright, and Ziesler [15], as well as the work of Olevskii [12].

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