A REMARK ON THE MAHLER CONJECTURE: LOCAL MINIMALITY OF THE UNIT CUBE

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Abstract. We prove that the unit cube B^n_∞ is a strict local minimizer for the Mahler volume product $vol_n(K)vol_n(K^*)$ in the class of origin symmetric convex bodies endowed with the Banach-Mazur distance.

1. Introduction

In 1939 Mahler [Ma] asked the following question. Let $K \subset \mathbb{R}^n$, $n \geqslant 2$, be a convex origin-symmetric body and let

$$K^* := \{\xi \in \mathbb{R}^n : x \cdot \xi \leqslant 1 \ \forall x \in K\}$$

be its polar body. Define $\mathcal{P}(K) = \mathrm{vol}_n(K)\mathrm{vol}_n(K^*)$. Is it true that we always have

$$\mathcal{P}(K) \geqslant \mathcal{P}(B^n_\infty),$$

where $B^n_\infty = \{x \in \mathbb{R}^n : |x_i| \leqslant 1, 1 \leqslant i \leqslant n\}$?

Mahler himself proved in [Ma] that the answer is affirmative when $n = 2$. There are several other proofs of the two-dimensional result, see for example the proof of M. Meyer, [Me2], but the question is still open even in the three-dimensional case.

In the n-dimensional case, the conjecture has been verified for some special classes of bodies, namely, for bodies that are unit balls of Banach spaces with 1-unconditional bases, [SR], [R2], [Me1], and for zonoids, [R1], [GMR].

Bourgain and Milman [BM] (see also [Pi]) proved the inequality

$$\mathcal{P}(K)^{1/n} \geqslant c\mathcal{P}(B^n_\infty)^{1/n},$$

with some constant $c > 0$ independent of n. The best known constant $c = \pi/4$ is due to Kuperberg [Ku].

Note that the exact upper bound for $\mathcal{P}(K)$ is known:

$$\mathcal{P}(K) \leqslant \mathcal{P}(B^n_2),$$

where B^n_2 is the n-dimensional Euclidean unit ball. This bound was proved by Santalo [Sa]. In [Pe] and [MeP] it was shown that the equality holds only if K is an ellipsoid.

Let $d_{BM}(K,L) = \inf\{b/a : \exists T \in GL(n) \text{ such that } aK \subseteq TL \subseteq bK\}$ be the Banach-Mazur multiplicative distance between bodies $K, L \subset \mathbb{R}^n$. In this paper we prove the following result.

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Theorem. *Let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Then*

$$\mathcal{P}(K) \geqslant \mathcal{P}(B_\infty^n),$$

provided that $d_{BM}(K, B_\infty^n) \leqslant 1+\delta$, and $\delta = \delta(n) > 0$ is small enough. Moreover, the equality holds only if $d_{BM}(K, B_\infty^n) = 1$, i.e., if K is a parallelepiped.

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Notation. Given a set $F \subset \mathbb{R}^n$, we define $\mathrm{af}(F)$ to be the affine subspace of the minimal dimension containing F, and $l(F)$ to be the linear subspace parallel to $\mathrm{af}(F)$ of the same dimension. The boundary of a convex body K is denoted by ∂K. For a given set $P \subset \mathbb{R}^n$, we write $P^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0, \forall y \in P\}$. Let $\mathcal{F}$ be the set of all faces F of all dimensions of the cube B_∞^n. We denote by c_F the center of a face $F \in \mathcal{F}$. We also denote $B_p^n = \{x \in \mathbb{R}^n : \sum_i |x_i|^p \leqslant 1\}$.

By C and c (with various indices and superscripts) we denote large and small positive constants respectively that may change from line to line and may depend on the dimension n, but on nothing else.

2. Description of the proof

The first difficulty in proving local minimality of the unit cube is that there are plenty of small perturbations with the same volume product, namely all close parallelepipeds. We overcome this difficulty by choosing a "canonical representative" in each class of affinely equivalent convex bodies. More precisely, we consider only the bodies K for which the unit cube is a parallelepiped of the least volume containing K. In addition to taking care of all close parallelepipeds, it allows us to fix $2n$ points on the boundary of K and K^* (the centers of the $(n-1)$-dimensional faces of B_∞^n). Our next step is to choose several additional points on the boundary of K and K^* and to construct two (not necessarily convex) polytopes $P \subset K$ and $Q \subset K^*$ such that

$$\mathrm{vol}_n(P)\mathrm{vol}_n(Q) \geqslant \mathcal{P}(B_\infty^n) - C\delta^2,$$

where δ is the least positive number for which $(1-\delta)B_\infty^n \subset K$. We conclude that B_∞^n is a lower semi-stationary point for the volume product functional $\mathcal{P}$. This means that the perturbation of B_∞^n by δ in the Banach-Mazur distance may result in decreasing the product volume only by δ^2, i.e., in the second order rather than in the first. Our last step is to show that either K contains a point outside $(1+c\delta)P$ or K^* contains a point outside $(1+c\delta)Q$ for some small positive c. This allows us to conclude that $\mathcal{P}(K)$ exceeds $\mathrm{vol}_n(P)\mathrm{vol}_n(Q)$ by at least $c\delta$ and get the final estimate

$$\mathcal{P}(K) \geqslant \mathcal{P}(B_\infty^n) + c\delta - C\delta^2$$

from which the strict local minimality follows immediately.

It is worth mentioning that the first part of the proof (lower semi-stationarity) works equally well for some other polytopes, for example, for the regular icosahedron and dodecahedron in $\mathbb{R}^3$. This indicates that the widely discussed idea to prove the Mahler conjecture by creating some kind of "gradient flow" on the class of convex bodies with respect to the volume product functional may be harder to realize than it seems.

3. Auxiliary results

Note that $\mathcal{P}(TK) = \mathcal{P}(K)$ for all $T \in GL(n)$. We will use this fact for choosing a canonical position for K.

Lemma 1. *Let P be a parallelepiped of minimal volume containing a convex origin-symmetric body K (note that every such parallelepiped is origin symmetric as well). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $P = TB^n_\infty$. Then $T^{-1}K \subset B^n_\infty$ and $\pm e_j \in \partial T^{-1}K$, $j = 1, ..., n$.*

Proof. Note that B^n_∞ is a parallelepiped of minimal volume containing $T^{-1}K$. If $e_j \notin T^{-1}K$, then there exists an affine hyperplane $H \ni e_j$ such that $H \cap T^{-1}K = \varnothing$. Note that the volume of the parallelepiped bounded by $H, -H$, and the affine hyperplanes $\{x : x \cdot e_i = \pm 1\}$, $i \neq j$, equals $\mathrm{vol}_n(B^n_\infty)$, and that this parallelepiped still contains K. But then we can shift H and $-H$ towards K a little bit and a get a new parallelepiped of smaller volume containing K. $\square$

We shall need the following simple technical lemma.

Lemma 2. *Let $P \subset \mathbb{R}^n$ be a star-shaped (with respect to the origin) polytope such that every $(n-1)$-dimensional face F of P has area at least A and satisfies $\mathrm{dist}(\mathrm{af}(F), 0) \geqslant r$, where dist denotes the Euclidean distance. Let $x \notin (1+\delta)P$ for some $\delta > 0$. Then*

$$\mathrm{vol}_n(\mathrm{conv}(P, x)) \geqslant \mathrm{vol}_n(P) + \frac{\delta r A}{n}.$$

Proof. Let $y = \partial P \cap [0, x]$. Let F be a face of P containing y. Then $\mathrm{conv}(P, x) \setminus P$ contains the pyramid with base F and apex x. The assumptions of the lemma imply that the height of this pyramid is at least $\delta\,\mathrm{dist}(\mathrm{af}(F), 0) \geqslant \delta r$, so its volume is at least $\frac{\delta r A}{n}$. $\square$

If K is sufficiently close to B^n_∞, then K is also close to any parallelepiped of minimal volume containing K.

Lemma 3. *Let K be a convex body satisfying*

$$(1-\delta)B^n_\infty \subset K \subset B^n_\infty.$$

Then there exists a constant C and a linear operator T such that

$$(1-C\delta)B^n_\infty \subset T^{-1}K \subset B^n_\infty,$$

and $\pm e_i \in T^{-1}K$.

Proof. Let as before $P = TB^n_\infty$ be a parallelepiped of minimal volume containing K. Note that $\mathrm{vol}_n(P) \leqslant 2^n$. On the other hand, if $x \in P \setminus (1+\kappa)(1-\delta)B^n_\infty$, then, by Lemma 2,

$$\mathrm{vol}_n(P) \geqslant 2^n(1-\delta)^n + \kappa\frac{2^{n-1}}{n}(1-\delta)^n.$$

The right hand side is greater than 2^n if $\kappa > \kappa_0 = 2n((1-\delta)^{-n}-1)$. Thus, $P \subset (1+\kappa_0)(1-\delta)B^n_\infty$, and thereby $(1-\kappa_0)P \subset (1-\delta)B^n_\infty \subset K$. It remains to note that $\kappa_0 \leqslant 4n^2\delta$ for sufficiently small $\delta > 0$. □

Thus, replacing K by its suitable linear image we may assume everywhere below that $K \subset B^n_\infty$, $\pm e_j \in \partial K$, $j = 1, \dots, n$. Let $\delta > 0$ be the minimal number such that $(1-\delta)B^n_\infty \subset K$.

4. Computation of the kernel of the differential of the volume function

Choose some numbers $a_k > 0$, $k = 0, \dots, n-1$, and define the polytope Q_0 as the union of the simplices

$$S_{\mathbb{F}} = \mathrm{conv}(0, a_0 c_{F_0}, a_1 c_{F_1}, \dots, a_{n-1} c_{F_{n-1}}),$$

where $\mathbb{F} = \{F_0, \dots, F_{n-1}\}$ runs over all flags ($F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1}, \mathrm{dim} F_j = j$) of faces of the unit cube.

Choose now some points x_F close to $x_F^0 = a_{\mathrm{dim}F} c_F$ and consider the polytope Q defined in the same way using the points x_F. Consider the function $g(\{x_F\}_{F\in\mathcal{F}}) = \mathrm{vol}_n(Q)$. It is just a polynomial of degree n of the coordinates of x_F, so it is infinitely smooth.

Lemma 4. *If $\Delta x_F \in \mathbb{R}^n$, $\Delta x_F \perp c_F$ for all F, then $\{\Delta x_F\} \in \mathrm{Ker} D_{\{x_F^0\}} g$, where $D_X g$ is the differential of g at the point X.*

Proof. Since the kernel of the differential is a linear space, it suffices to check this for the vectors $\{\Delta x_F\}$ in which only one $\Delta x_{\widetilde{F}} \neq 0$. Due to symmetry, we may assume that $c_{\widetilde{F}} = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k})$. The space orthogonal to $c_{\widetilde{F}}$ is then generated by the vectors e_j, $j > k$ and $e_i - e_j$, $1 \leqslant i < j \leqslant k$. Note now that the polytopes Q^+ and Q^- built on the points x_F^0, $F \neq \widetilde{F}$, and $x^0_{\widetilde{F}} \pm h e_j$, where $j > k$, are symmetric with respect to the symmetry $e_j \to -e_j$, so their volumes are the same. On the other hand, the difference of their volumes in the first order is $2h D_{\{x_F^0\}} g(\{0, \dots, e_j, \dots, 0\})$, where e_j stands in the position corresponding to $\widetilde{F} \in \mathcal{F}$. Thus,

$$D_{\{x_F^0\}} g(\{0, \dots, e_j, \dots, 0\}) = 0.$$

To prove the equality $D_{\{x_F^0\}} g(\{0, \dots, e_i - e_j, \dots, 0\}) = 0$, consider Q' and Q'' built using the points $x_F = x_F^0$, $F \neq \widetilde{F}$ and $x_{\widetilde{F}} = x^0_{\widetilde{F}} + h e_i$ or $x_{\widetilde{F}} = x^0_{\widetilde{F}} + h e_j$ respectively. They are also symmetric with respect to the symmetry $e_i \leftrightarrow e_j$ and the difference of their volumes in the first order equals $h D_{\{x_F^0\}} g(\{0, \dots, e_i - e_j, \dots, 0\})$. □

Below we shall also need the following elementary observation from real analysis.

Lemma 5. *Let $g(X)$ be a smooth function on $\mathbb{R}^N$ and $X_0 \in \mathbb{R}^N$. There exists a constant* Const *depending on g and X_0 such that for all sufficiently small $\delta > 0$ and all $X_1, X_2 \in \mathbb{R}^N$ satisfying*

$$\|X_1 - X_0\|, \|X_2 - X_0\| \leqslant \delta \text{ and } X_1 - X_2 \in \mathrm{Ker} D_{X_0} g,$$

one has $|g(X_1) - g(X_2)| \leqslant \mathrm{Const}\,\delta^2$.

Proof. Using the Taylor formula, we get

$$g(X_j) = g(X_0) + (D_{X_0}g)\,(X_j - X_0) + O(\delta^2), \text{ where } j = 1, 2.$$

Subtracting these two identities, we obtain

$$g(X_1) - g(X_2) = (D_{X_0}g)\,(X_1 - X_2) + O(\delta^2) = O(\delta^2),$$

because $(D_{X_0}g)\,(X_1 - X_2) = 0$. □

Let $P \subset \mathbb{R}^n$ be a convex polytope. For a face F of P, we define its dual face F^* of P^* by $F^* = \{y \in P^* : x \cdot y = 1 \text{ for all } x \in F\}$ (see Chapter 3.4 in [Gr]).

Lemma 6. *Let P be a convex polytope such that 0 is in the interior of P. Let P^* be its dual polytope. Choose some pair of dual faces F and F^* of P and P^* respectively and some points $x \in F$, $x^* \in F^*$ in the relative interiors of F and F^*. Assume that K is a convex body satisfying $(1-\delta)P \subset K \subset P$. Then there exists a pair of points $y \in \partial K$ and $y^* \in \partial K^*$ such that $y \cdot y^* = 1$ and $\|y - x\|, \|y^* - x^*\| \leqslant C\delta$, where $C > 0$ does not depend on K or δ, but may depend on P, P^*, F, F^*, x and x^*.*

Proof. Since $x \cdot x^* = 1 > 0$, there exists a self-adjoint positive definite linear operator A such that $Ax = x^*$. This operator can be chosen as follows: Let L be a 2-dimensional plane through the origin containing both x and x^*. A will act identically on $L^\perp$. To define its action on L, choose an orthogonal basis e_1, e_2 in L such that $e_1 = x$ and put

$$A\big|_L = \begin{pmatrix} a & b \\ b & a' \end{pmatrix},$$

where $x^* = ae_1 + be_2$ and $a' > 0$ is chosen so large that $aa' > b^2$.

We will use below the following simple orthogonality relations:

(1) $x \perp l(F^*)$.
(2) $x^* \perp l(F)$.
(3) $l(F) \perp l(F^*)$.
(4) $[A^{-1}l(F^*)]^\perp = \operatorname{span}[x^*, Al(F)]$ and $[Al(F)]^\perp = \operatorname{span}[x, A^{-1}l(F^*)]$.
(5) $(x^*)^\perp \cap \operatorname{span}(x, A^{-1}l(F^*)) = A^{-1}l(F^*)$.

$(1), (2)$ and (3) follow directly from the definition of F^* (see Chapter 3.4 in [Gr]). Let us first prove (4). Since $l(F) \perp l(F^*)$ and A is self-adjoint, we also have $Al(F) \perp A^{-1}l(F^*)$. Also, since $x \perp l(F^*)$, we have $x^* = Ax \perp A^{-1}l(F^*)$. Thus $\operatorname{span}(x^*, Al(F)) \subset [A^{-1}l(F^*)]^\perp$. On the other hand, $x \notin l(F)$, so $x^* = Ax \notin Al(F)$ and

$$\dim(\operatorname{span}(x^*, Al(F))) = 1 + \dim F = n - \dim F^* = n - \dim A^{-1}l(F^*),$$

so $A^{-1}l(F^*)^\perp$ can not be wider than $\operatorname{span}(x^*, Al(F))$. Similarly,

$$[Al(F)]^\perp = \operatorname{span}\left[x, A^{-1}l(F^*)\right].$$

To prove (5), we first note that $A^{-1}l(F^*) \perp x^*$ (see (4)). Since $x^* \cdot x = 1 \neq 0$, $(x^*)^\perp \cap \operatorname{span}(x, A^{-1}l(F^*))$ is a subspace of codimension 1 in $\operatorname{span}(x, A^{-1}l(F^*))$, so it cannot be wider than $A^{-1}l(F^*)$.

Let $\widetilde{K} = K \cap \mathrm{span}(x, A^{-1}l(F^*))$ and let $y \in \widetilde{K}$ maximize $y \cdot x^*$. Then $y \in \partial K$ and a tangent plane to K at y contains an affine plane parallel to $(x^*)^{\perp} \cap \mathrm{span}(x, A^{-1}l(F^*)) = A^{-1}l(F^*)$. Therefore, there exists $y^* \in \partial K^* \cap [A^{-1}l(F^*)]^{\perp} = \partial K^* \cap \mathrm{span}(x^*, Al(F))$ such that $y \cdot y^* = 1$.

Now let $y = \alpha x + h$ and $y^* = \alpha^* x^* + h^*$, where $h \in A^{-1}l(F^*)$ and $h^* \in Al(F)$. Note that $y \cdot x^* = \alpha$, so by maximality of y,

$$\alpha = (y, x^*) > (0, x^*) = 0.$$

Also $y \cdot y^* = \alpha\alpha^* = 1$, so $\alpha^* > 0$. Let $\rho > 0$ be such that $B(x,\rho) \cap \mathrm{af}(F) \subset F$ and $B(x^*,\rho) \cap \mathrm{af}(F^*) \subset F^*$ where $B(z,t)$ is the Euclidean ball of radius t centered at z. Since $y \in \partial K$ and

$$K^* \supset P^* \supset F^* \ni x^* + \frac{\rho A h}{\|Ah\|},$$

we have

$$1 \geqslant y \cdot \left(x^* + \frac{\rho A h}{\|Ah\|} \right) = \alpha + \frac{\rho A h \cdot h}{\|Ah\|} \geqslant \alpha + \rho' \|h\|, \text{ where } \rho' = \frac{\rho}{\|A\| \|A^{-1}\|}.$$

Since $y^* \in \partial K^*$ and

$$K \supset (1-\delta) P \supset (1-\delta) F \ni (1-\delta) \left[x + \frac{\rho A^{-1} h^*}{\|A^{-1} h^*\|} \right],$$

we have

$$\left((1-\delta) \left[x + \frac{\rho A^{-1} h^*}{\|A^{-1} h^*\|} \right] \right) \cdot y^* \leqslant 1$$

and

$$(1-\delta)^{-1} \geqslant \left[x + \frac{\rho A^{-1} h^*}{\|A^{-1} h^*\|} \right] \cdot y^* = \alpha^* + \frac{\rho A^{-1} h^* \cdot h^*}{\|A^{-1} h^*\|} \geqslant \alpha^* + \rho' \|h^*\|.$$

Thus $\alpha \leqslant 1$ and $\alpha^* \leqslant 1/(1-\delta)$, which, together with $\alpha\alpha^* = 1$, gives $\alpha \geqslant 1-\delta$ and $\alpha^* \geqslant 1$. Hence $\rho'\|h\| \leqslant \delta$, $\rho'\|h^*\| \leqslant \frac{1}{1-\delta} - 1$ and, thereby, $\|y - x\|, \|y^* - x^*\| \leqslant C\delta$. $\square$

Remark: Below (in Section 5) we will need Lemma 6 only for the case when x and x^* are collinear. In this case we can choose A to be a pure homothety and get the points $y = \alpha x + h$ and $y^* = \alpha^* x^* + h^*$, with $h \in l(F^*)$ and $h^* \in l(F)$.

Now define $c_F^* = \frac{1}{n - \mathrm{dim} F} c_F$. Choose positive numbers α_F and α_F^* satisfying $\alpha_F \alpha_F^* = 1$ and put $y_F = \alpha_F c_F$, $y_F^* = \alpha_F^* c_F^*$.

Let $Q = \cup_{\mathbb{F}} S_{\mathbb{F}}(Q)$ and $Q' = \cup_{\mathbb{F}} S_{\mathbb{F}}(Q')$, where

$$S_{\mathbb{F}}(Q) = \mathrm{conv}(0, y_{F_0}, y_{F_1}, \dots, y_{F_{n-1}}) \text{ and } S_{\mathbb{F}}(Q') = \mathrm{conv}(0, y^*_{F_0}, y^*_{F_1}, \dots, y^*_{F_{n-1}})$$

and $\mathbb{F}$ runs over all flags $\mathbb{F} = \{F_0, \dots, F_{n-1}\}$ of faces of B_∞^n.

Lemma 7.

$$\mathrm{vol}_n(Q)\mathrm{vol}_n(Q') \geqslant \mathcal{P}(B_\infty^n).$$

Proof. For every flag $\mathbb{F} = \{F_0, \dots, F_{n-1}\}$,

$$\mathrm{vol}_n(S_{\mathbb{F}}(Q)) = \mathrm{vol}_n(S_{\mathbb{F}}(B_\infty^n)) \prod_{j=0}^{n-1} \alpha_{F_j}, \text{ where } S_{\mathbb{F}}(B_\infty^n) = \mathrm{conv}(0, c_{F_0}, c_{F_1}, \dots, c_{F_{n-1}}),$$

and

$$\mathrm{vol}_n(S_{\mathbb{F}}(Q')) = \mathrm{vol}_n(S_{\mathbb{F}}(B_1^n)) \prod_{j=0}^{n-1} \alpha^*_{F_j}, \text{ where } S_{\mathbb{F}}(B_1^n) = \mathrm{conv}(0, c^*_{F_0}, c^*_{F_1}, \dots, c^*_{F_{n-1}}).$$

Hence,

$$\mathrm{vol}_n(S_{\mathbb{F}}(Q))\mathrm{vol}_n(S_{\mathbb{F}}(Q')) = \mathrm{vol}_n(S_{\mathbb{F}}(B_\infty^n))\mathrm{vol}_n(S_{\mathbb{F}}(B_1^n)).$$

The factors on the right hand side do not depend on the flag $\mathbb{F}$. Thus,

$$\mathrm{vol}_n(Q)\mathrm{vol}_n(Q') = \sum_{\mathbb{F}} \mathrm{vol}_n(S_{\mathbb{F}}(Q)) \sum_{\mathbb{F}} \mathrm{vol}_n(S_{\mathbb{F}}(Q'))$$

$$\geqslant \left(\sum_{\mathbb{F}} \sqrt{\mathrm{vol}_n(S_{\mathbb{F}}(Q))\mathrm{vol}_n(S_{\mathbb{F}}(Q'))} \right)^2 = \left(\sum_{\mathbb{F}} \sqrt{\mathrm{vol}_n(S_{\mathbb{F}}(B_\infty^n))\mathrm{vol}_n(S_{\mathbb{F}}(B_1^n))} \right)^2$$

$$= \sum_{\mathbb{F}} \mathrm{vol}_n(S_{\mathbb{F}}(B_\infty^n)) \sum_{\mathbb{F}} \mathrm{vol}_n(S_{\mathbb{F}}(B_1^n)) = \mathrm{vol}_n(B_\infty^n)\mathrm{vol}_n(B_1^n) = \mathcal{P}(B_\infty^n).$$

□

5. Lower stationarity of B_∞^n

Now apply Lemma 6 to B_∞^n and B_1^n and the points $c_F \in F$ and $c^*_F = \frac{1}{n - \mathrm{dim} F} c_F \in F^*$, where F^* is the face of B_1^n dual to F. Since c_F and c^*_F are collinear, we get points

$$x_F = \alpha_F c_F + h_F \in \partial K \text{ and } x^*_F = \alpha^*_F c^*_F + h^*_F \in \partial K^*,$$

where $\alpha_F \alpha^*_F = 1$, $h_F \in l(F^*)$, $h^*_F \in l(F)$ and $|\alpha_F - 1|, |\alpha^*_F - 1|, \|h_F\|, \|h^*_F\| \leqslant C\delta$.

Since $\pm e_j \in \partial K$ and $\pm e_j \in \partial K^*$, we can choose $x_F = y_F = x^*_F = y^*_F = c_F = c^*_F$ when $\mathrm{dim} F = n - 1$.

Put $y_F = \alpha_F c_F$ and $y^*_F = \alpha^*_F c^*_F$, and consider the polytopes

$$P = \cup_{\mathbb{F}} \mathrm{conv}(0, x_{F_0}, \dots, x_{F_{n-1}}) \text{ and } P' = \cup_{\mathbb{F}} \mathrm{conv}(0, x^*_{F_0}, \dots, x^*_{F_{n-1}}),$$

$$Q = \cup_{\mathbb{F}} \mathrm{conv}(0, y_{F_0}, \dots, y_{F_{n-1}}) \text{ and } Q' = \cup_{\mathbb{F}} \mathrm{conv}(0, y^*_{F_0}, \dots, y^*_{F_{n-1}}).$$

Note that $x_F - y_F = h_F$, $x^*_F - y^*_F = h^*_F$ and $h_F, h^*_F \perp c_F$.

Thus by Lemmata 4, 5.

$$|\mathrm{vol}_n(P) - \mathrm{vol}_n(Q)| \leqslant C\delta^2 \text{ and } |\mathrm{vol}_n(P') - \mathrm{vol}_n(Q')| \leqslant C\delta^2,$$

whence

$$\mathrm{vol}_n(P)\mathrm{vol}_n(P') \geqslant \mathrm{vol}_n(Q)\mathrm{vol}_n(Q') - C\delta^2 \geqslant \mathcal{P}(B_\infty^n) - C\delta^2,$$

where the last inequality follows from Lemma 7.

Since $K \supset P$ and $K^* \supset P'$, it remains to show that for some $c > 0$, either $K \not\subset (1+c\delta)P$, or $K^* \not\subset (1+c\delta)P'$. Then, by Lemma 2, either $\mathrm{vol}_n(K) \geqslant \mathrm{vol}_n(P) + c'\delta$, or $\mathrm{vol}_n(K^*) \geqslant \mathrm{vol}_n(P') + c'\delta$. This yields

$$\mathcal{P}(K) \geqslant \mathcal{P}(B_\infty^n) + c''\delta - C\delta^2 > \mathcal{P}(B_\infty^n),$$

provided that $\delta > 0$ is small enough.

6. The conclusion of the proof

Note that at least one of the coordinates of one of the $x_{\widetilde{F}}$ with $\dim\widetilde{F} = 0$ is at most $1-\delta$. Indeed, assume that all coordinates are greater than $(1-\delta')$ in absolute value with some $\delta' < \delta$. Define $D = \mathrm{conv}\{x_F : F \in \mathcal{F}, \dim F = 0\} \subset K$. Let $z \in D^*$. Choose F so that $(x_F)_j z_j \geqslant 0$ for all $j = 1, \dots, n$. Then

$$1 \geqslant x_F \cdot z \geqslant (1-\delta') \sum_j |z_j|.$$

Thus, $D^* \subset (1-\delta')^{-1} B_1^n$ and $D \supset (1-\delta')B_\infty^n$, contradicting the minimality of δ.

Due to symmetry, we may assume without loss of generality that $\widetilde{F} = \{(1, \dots, 1)\}$ and that $(x_{\widetilde{F}})_1 \leqslant 1-\delta$. Assume that $K \subset (1+c\delta)P$. Consider the point $\tilde{x} = (1-\delta, c'\delta, \dots, c'\delta)$, where $c' = 1/(n-\frac{5}{4})$. Then $\tilde{x} \in (1-c''\delta)P^*$, where $c'' = 1/(4n-5)$. Indeed, it is enough to check that $\tilde{x} \cdot x_F \leqslant 1 - c''\delta$ for all vertices x_F of P. If $F \neq \{(1, \dots, 1)\}$, then all coordinates of x_F do not exceed 1 and at least one does not exceed $1/2$. Thus, if δ is small enough, we get

$$\tilde{x} \cdot x_F \leqslant (1-\delta) + (n-2)c'\delta + \frac{c'\delta}{2} = 1 - \delta + (n-\tfrac{3}{2})c'\delta = 1 - c''\delta.$$

If $F = \{(1, \dots, 1)\}$, then

$$\tilde{x} \cdot x_F \leqslant (1-\delta)^2 + (n-1)c'\delta = 1 - 2\delta + \frac{n-1}{n-\frac{5}{4}}\delta + \delta^2 \leqslant 1 - 2\delta + \frac{4}{3}\delta + \delta^2 \leqslant 1 - c''\delta,$$

provided that $\delta > 0$ is small enough. Therefore if $c < c''$, we get $\tilde{x} \in \frac{1}{1+c\delta}P^* \subset K^*$.

Now note that for every $x \in P'$, we have

$$|x_1| + (1 - C'\delta)\sum_{j \geqslant 2} |x_j| \leqslant 1,$$

provided C' is chosen large enough. Indeed, again it is enough to check this for the vertices x_F^* of P'. If $c_F \neq (\pm 1, 0, \dots, 0)$ we have $\sum_{j \geqslant 2} |(x_F^*)_j| \geqslant 1/3$, so

$$|(x_F^*)_1| + (1-C'\delta)\sum_{j \geqslant 2} |(x_F^*)_j| \leqslant \sum_{j \geqslant 1} |(x_F^*)_j| - C'\delta \sum_{j \geqslant 2} |(x_F^*)_j| \leqslant 1 + nC\delta - \frac{C'\delta}{3} \leqslant 1,$$

provided that $C' \geqslant 3nC$, where C is the constant such that $\|x_F^* - c_F^*\| \leqslant C\delta$. If $c_F = (\pm 1, 0 \dots, 0)$, then $x_F = \pm e_1$ and the inequality is trivial.

Now it remains to note that

$$|\tilde{x}_1| + (1-C'\delta)\sum_{j \geqslant 2} |\tilde{x}_j| = 1 - \delta + (1-C'\delta)(n-1)c'\delta = 1 + c''\delta - C'(n-1)c'\delta^2 > 1 + c\delta,$$

provided that $c < c''/2$ and δ is small enough, whence $\tilde{x} \notin (1+c\delta)P'$.

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