FINE APPROXIMATION OF CONVEX BODIES BY POLYTOPES

MÁRTON NASZÓDI, FEDOR NAZAROV, AND DMITRY RYABOGIN

Abstract. We prove that for every convex body K with the center of mass at the origin and every $\varepsilon \in \left(0, \frac{1}{2}\right)$, there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.

1. Introduction and main result

A *convex body* in $\mathbb{R}^d$ is a compact convex set with non-empty interior. Our goal is to prove the following theorem.

Theorem. *Let K be a convex body in $\mathbb{R}^d$ with the center of mass at the origin, and let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then there exists a convex polytope P with at most $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ vertices such that $(1-\varepsilon)K \subset P \subset K$.*

This result improves the 2012 theorem of Barvinok [B] by removing the symmetry assumption and the extraneous $(\log \frac{1}{\varepsilon})^d$ factor. Our approach uses a mixture of geometric and probabilistic tools.

We refer the reader to the surveys of Bronshtein [Br] and Gruber [Gr] for the discussion of the history of the problem. Unfortunately, we will have to rely upon two non-trivial classical results (Blaschke-Santaló inequality and its reverse), which makes this paper a bit less reader-friendly than we would like it to be despite our best efforts to provide well-written and easily accessible references for all statements that we use without a proof.

2. Outline of the proof

Without loss of generality, we may assume that K has smooth boundary, in particular, K has a unique supporting hyperplane at each

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boundary point. Our task is to find a finite set of points $Y \subset \partial K$ such that $P = \operatorname{conv} Y$ satisfies $(1-\varepsilon)K \subset P$. By duality, this is equivalent to the requirement that every cap $S(x,\varepsilon) = \{y \in \partial K : \langle y, \nu_x\rangle \geqslant (1-\varepsilon)\langle x, \nu_x\rangle\}$, where $x \in \partial K$ and ν_x is the outer unit normal to ∂K at x, contains at least one point of Y.

The key idea is to construct a probability measure μ on ∂K such that for every $x \in \partial K, \varepsilon \in \left(0, \frac{1}{2}\right)$, we have $\mu(S(x,\varepsilon)) \geqslant p\varepsilon^{\frac{d-1}{2}}$ with some $p = e^{O(d)}$ depending on d only.

Since there are infinitely many caps, our next aim is to choose an appropriate finite net $X \subset \partial K$ of cardinality $C(d)\varepsilon^{-\frac{d-1}{2}}$ such that the condition $S(x, \frac{\varepsilon}{2}) \cap Y \neq \varnothing$ for all $x \in X$ implies that $S(x,\varepsilon) \cap Y \neq \varnothing$ for all $x \in \partial K$. Given such a net, we will be able to apply a general combinatorial result essentially due to Rogers to construct the desired set Y of cardinality approximately $\log C(d) p^{-1}\varepsilon^{-\frac{d-1}{2}}$, which will be still $e^{O(d)}\varepsilon^{-\frac{d-1}{2}}$ as long as $C(d)$ is at most double exponential in d.

A natural net to try is the Bronshtein–Ivanov net (see [BI]), which allows one to approximate a point $x \in \partial K$ and the corresponding outer unit normal ν_x by a point in the net and its outer unit normal simultaneously. Unfortunately, it works only for uniformly 2-convex bodies, i.e., the bodies that can be touched by an outer sphere of fixed controllable radius at every boundary point.

So, the last step will be to show that the task of approximating an arbitrary convex body K can be reduced to that of approximating a certain uniformly 2-convex body associated with K.

In the exposition, these steps are presented in reverse. We start with constructing the associated uniformly 2-convex body (Sections 3, 4, 5). Then we build the Bronshtein-Ivanov net X of appropriate mesh and cardinality, and check that it is, indeed, enough to consider the caps $S(x, \frac{\varepsilon}{2}), x \in X$ (Sections 6, 7, 8). Finally, we construct the probability measure μ and finish the proof of the theorem (Sections 9, 10).

3. Standard position

Since the problem is invariant under linear transformations, we can always assume that our body K is in some "standard position". The exact notion of the standard position to use is not very important as long as it guarantees that $B \subset K \subset d^2 B$, say, where B is the unit ball in $\mathbb{R}^d$ centered at the origin.

One possibility is to make a linear transformation such that John's ellipsoid (see [Ba], Lecture 3) of the centrally-symmetric convex body $L = K \cap -K$ is the unit ball, so $B \subset L \subset \sqrt{d}B$ and, since $K \supset -\frac{1}{d}K$ (see [BF], page 57), it follows that $B \subset K \subset d\sqrt{d}B$.

4. The function φ_δ and the mapping Φ_δ

Fix $\delta \in \left(0, \frac{1}{2}\right)$. For $r \geqslant 0$, define $\varphi_\delta(r)$ as the positive root of the equation $\varphi + \delta r^2 \varphi^2 = 1$. Put $\Phi_\delta(x) = x\varphi_\delta(|x|)$, $x \in \mathbb{R}^d$.

Lemma 1. *The function φ_δ is a decreasing smooth function on $[0, +\infty)$; $r \mapsto r\varphi_\delta(r)$ is an increasing function mapping $[0, +\infty)$ to $[0, \delta^{-\frac{1}{2}})$; Φ_δ is a diffeomorphism of $\mathbb{R}^d$ onto the open ball $\delta^{-\frac{1}{2}} \operatorname{int} B$; if ν is a unit vector and $h > 0$, then the image $\Phi_\delta(H_{\nu,h})$ of the half-space $H_{\nu,h} = \{x : \langle x, \nu \rangle \leqslant h\}$ is the intersection of $\delta^{-\frac{1}{2}} \operatorname{int} B$ and the ball of radius $\sqrt{\frac{1}{4\delta^2 h^2} + \frac{1}{\delta}}$ centered at $-\frac{1}{2\delta h}\nu$ (see Figure 1).*

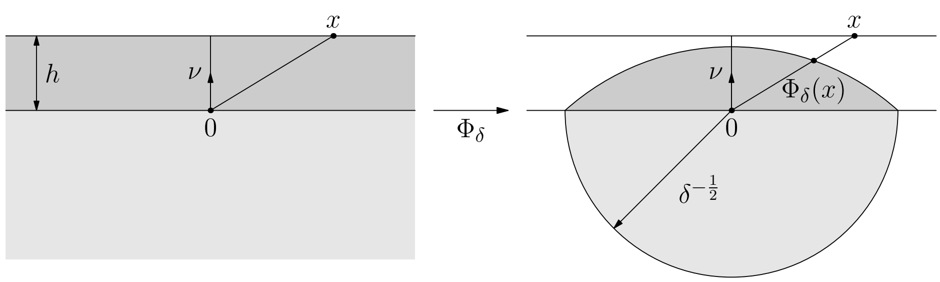

Figure 1. The mapping Φ_δ

Proof. The first statement is obvious. To show the second one, just notice that $r\varphi_\delta(r)$ is the positive root of $\frac{\psi}{r} + \delta\psi^2 = 1$ and, as $r \to \infty$, this root increases to $\delta^{-\frac{1}{2}}$. The third claim follows from the observation that the derivative of the mapping $r \mapsto r\varphi_\delta(r)$ is strictly positive and continuous on $[0, +\infty)$. To prove the last claim, observe that if $\langle x, \nu \rangle = h$, then

$$\left|\Phi_\delta(x) + \frac{1}{2\delta h}\nu\right|^2 = \left|x\varphi_\delta(|x|) + \frac{1}{2\delta h}\nu\right|^2 = \\ |x|^2\varphi_\delta(|x|)^2 + \frac{\varphi_\delta(|x|)}{\delta} + \frac{1}{4\delta^2 h^2} = \frac{1}{4\delta^2 h^2} + \frac{1}{\delta}$$

by the definition of φ_δ. □

It follows that for every convex body K containing the origin, $\Phi_\delta(K)$ is also convex. Since for every interval $I_x = \{rx : 0 \leqslant r \leqslant 1\}$, $x \in \mathbb{R}^d$, we have $\Phi_\delta(I_x) \subset I_x$, the image $\Phi_\delta(K)$ is contained in K. Moreover, if $B \subset K$, then $\Phi_\delta(K)$ is the intersection of balls of radii not exceeding $\sqrt{\frac{1}{4\delta^2} + \frac{1}{\delta}} \leqslant \frac{1}{\delta}$. In particular, for every boundary point $x \in \partial\Phi_\delta(K)$, we can find a ball of radius $\frac{1}{\delta}$ containing $\Phi_\delta(K)$ whose boundary sphere touches $\Phi_\delta(K)$ at x.

5. From the approximation of $\Phi_\delta(K)$ to the approximation of K

Lemma 2. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Suppose that a convex body K satisfies $0 \in K \subset d^2 B$ and $\delta < \frac{1}{4d^4}$. If $Y \in \partial K$ is a finite set such that $\left(1-\frac{\varepsilon}{2}\right)\Phi_\delta(K) \subset \operatorname{conv}(\Phi_\delta(Y))$, then $(1-\varepsilon)K \subset \operatorname{conv}(Y)$.*

Proof. Note that the conditions of the lemma imply that $0 \in \operatorname{conv}(\Phi_\delta(Y))$. Since for every $y \in \mathbb{R}^d$, $\Phi_\delta(y)$ is a positive multiple of y, we conclude that $0 \in P = \operatorname{conv}(Y)$ as well, so $\Phi_\delta(P)$ is convex. Suppose that there exists $x \in K$ such that $(1-\varepsilon)x \notin P$. Then,

$$\Phi_\delta((1-\varepsilon)x) \notin \Phi_\delta(P) \supset \operatorname{conv}(\Phi_\delta(Y)).$$

However,

$$\Phi_\delta((1-\varepsilon)x) = (1-\varepsilon)\frac{\varphi_\delta((1-\varepsilon)|x|)}{\varphi_\delta(|x|)}\Phi_\delta(x).$$

Denoting $\eta_t = \varphi_\delta((1-t)|x|)$, $t \in [0,1]$, we have

$$\eta_\varepsilon + \delta(1-\varepsilon)^2|x|^2\eta_\varepsilon^2 = \eta_0 + \delta|x|^2\eta_0^2 = 1.$$

Since $\delta|x|^2\eta_\varepsilon^2 \geqslant \delta|x|^2\eta_0^2$ and $\delta\varepsilon^2|x|^2\eta_\varepsilon^2 \geqslant 0$, it follows that

$$\eta_\varepsilon(1-2\delta\varepsilon|x|^2\eta_\varepsilon) \leqslant \eta_0, \quad \text{so} \quad \frac{\eta_\varepsilon}{\eta_0} \leqslant \frac{1}{1-2\delta\varepsilon|x|^2\eta_\varepsilon}.$$

Since $\eta_\varepsilon \leqslant 1$ and $2\delta|x|^2 \leqslant 2\delta d^4 \leqslant \frac{1}{2}$, we get

$$(1-\varepsilon)\frac{\eta_\varepsilon}{\eta_0} \leqslant \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}} \leqslant 1 - \frac{\varepsilon}{2},$$

so $\left(1-\frac{\varepsilon}{2}\right)\Phi_\delta(x)$ cannot be contained in $\operatorname{conv}(\Phi_\delta(Y))$, which contradicts our assumption. $\square$

This lemma implies that an $\frac{\varepsilon}{2}$-approximation of $\Phi_\delta(K)$ yields an ε-approximation of K. Note also that $\Phi_\delta(K)$ is rather close to K. More precisely, if $0 \in K \subset d^2B$, we have $(1-\delta d^4)K \subset \Phi_\delta(K) \subset K$. The center of mass of $\Phi_\delta(K)$ may no longer be at the origin, of course, but the only non-trivial property of K we shall really use is the Santaló bound $\operatorname{vol}_d(K)\operatorname{vol}_d(K^\circ) \leqslant e^{O(d)}d^{-d}$, where

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \quad \text{for all} \quad x \in K\}$$

is the polar body of the convex body K. This bound holds for K because 0, being the center of mass of K, is, thereby, the Santaló point of K° (see Section 10 for details). For sufficiently small $\delta > 0$, it is inherited by $\Phi_\delta(K)$ just because $(\Phi_\delta(K))^\circ \subset (1-\delta d^4)^{-1}K^\circ$ and, thereby,

$$\operatorname{vol}_d(\Phi_\delta(K))\operatorname{vol}_d((\Phi_\delta(K))^\circ) \leqslant (1-\delta d^4)^{-d}\operatorname{vol}_d(K)\operatorname{vol}_d(K^\circ).$$

Choosing $\delta = \frac{1}{4d^5}$, we see that the body $\Phi_\delta(K)$ also satisfies the Santaló bound with only marginally worse constant. At last, if $B \subset K$, we have $\frac{1}{2}B \subset (1-\delta)B \subset \Phi_\delta(K)$.

Thus, replacing K by $\Phi_\delta(K)$ (and ε by $\frac{\varepsilon}{2}$) if necessary, from now on we can restrict ourselves to the class $\mathcal{K}_R$ of convex bodies K with smooth boundary such that $\frac{1}{2}B \subset K \subset d^2B$ and for every boundary point $x \in \partial K$, there exists a ball of fixed radius $R = 4d^5$ containing K whose boundary sphere touches K at x. Moreover, we can also assume that $\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ) \leqslant e^{O(d)}d^{-d}$.

6. The Bronshtein–Ivanov net

Let $\rho \in \left(0, \frac{1}{2}\right)$. Let K be a convex body with smooth boundary containing the origin and contained in d^2B. Consider the set S of points $\{x + \nu_x : x \in \partial K\}$, where ν_x is the outer unit normal to ∂K at x. Let $\{x_j + \nu_{x_j} : 1 \leqslant j \leqslant N\}$ be a maximal ρ-separated set in S, i.e., a set such that any two of its members are at distance at least ρ (see Figure 2). We will call the corresponding set $\{x_j : 1 \leqslant j \leqslant N\}$ a Bronshtein-Ivanov net of mesh ρ for the body K.

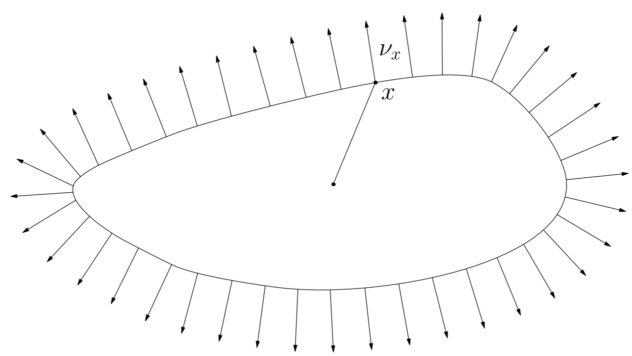

Figure 2. The Bronshtein-Ivanov net

Lemma 3. *We have $N \leqslant 2^d(d^2+3)^d\rho^{-d+1}$, and for every $x \in \partial K$, we can find j such that $|x - x_j|^2 + |\nu_x - \nu_{x_j}|^2 \leqslant \rho^2$.*

Proof. Let $x', x'' \in \partial K$ and let $\nu' = \nu_{x'}$, $\nu'' = \nu_{x''}$. Note that, by the convexity of K, we must have $\langle \nu', x' - x''\rangle \geqslant 0$, $\langle \nu'', x'' - x'\rangle \geqslant 0$. Hence, we always have

$$|x' + \nu' - x'' - \nu''|^2 = |x' - x''|^2 + |\nu' - \nu''|^2 + 2(\langle \nu', x' - x''\rangle + \langle \nu'', x'' - x'\rangle) \geqslant |x' - x''|^2 + |\nu' - \nu''|^2,$$

and the second conclusion of the lemma follows immediately from the definition of x_j.

Now assume that $s', s'' \geqslant 0$. Write

$$|x' + \nu' + s'\nu' - x'' - \nu'' - s''\nu''|^2 = |x' + \nu' - x'' - \nu''|^2 + \\ |s'\nu' - s''\nu''|^2 + 2s'\langle \nu', x' - x''\rangle + 2s''\langle \nu'', x'' - x'\rangle + \\ 2(s' + s'')(1 - \langle \nu', \nu''\rangle) \geqslant |x' + \nu' - x'' - \nu''|^2.$$

Thus, if the balls of radius $\frac{\rho}{2}$ centered at $x'+\nu'$ and $x''+\nu''$ are disjoint, so are the balls of radius $\frac{\rho}{2}$ centered at $x'+(1+s')\nu'$ and $x''+(1+s'')\nu''$. From here we conclude that the balls of radius $\frac{\rho}{2}$ centered at the points $x_j+(1+k\rho)\nu_{x_j}$, $0 \leqslant k \leqslant \frac{1}{\rho}$ are all disjoint (see Figure 3) and contained in $(d^2+3)B$.

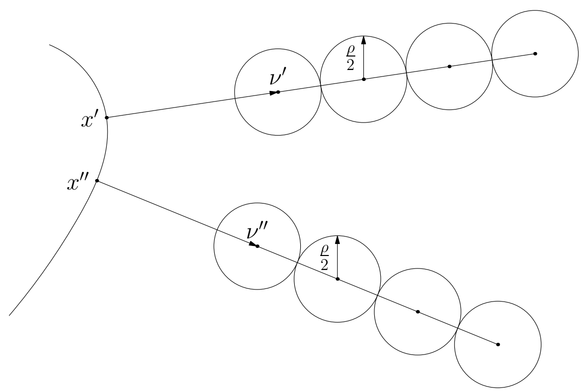

FIGURE 3. The disjoint balls

The total number of these balls is at least $\frac{N}{\rho}$ (for every point x_j in the net, there is a chain of at least $\frac{1}{\rho}$ balls corresponding to different values of k), whence $\frac{N}{\rho} \leqslant \left(\frac{d^2+3}{\frac{\rho}{2}}\right)^d$ and the desired bound for N follows. □

7. THE DISTANCE BOUND

The following lemma shows that ε-caps of convex bodies $K \in \mathcal{K}_R$ have small diameters.

Lemma 4. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Assume that $K \in \mathcal{K}_R$, $x \in \partial K$, and ν is the outer normal to ∂K at x. If $y \in S(x, \varepsilon)$, i.e., $y \in K$ and $\langle y, \nu\rangle \geqslant (1-\varepsilon)\langle x, \nu\rangle$, then $|y - x| \leqslant \sqrt{2R}\, d\sqrt{\varepsilon}$.*

Proof. Let Q be the ball of radius R containing K whose boundary sphere touches K at x. Then $y \in Q$ and ν is the outer unit normal to

Q at x, so Q is centered at $x - R\nu$. Note also that, since $0 \in K \subset d^2 B$, we have $0 \leqslant \langle x, \nu\rangle \leqslant d^2$. Now we have

$$R^2 \geqslant |y - x + R\nu|^2 = |y - x|^2 + 2R\langle y - x, \nu\rangle + R^2,$$

so

$$|y - x|^2 \leqslant 2R\langle x - y, \nu\rangle \leqslant 2R\varepsilon\langle x, \nu\rangle \leqslant 2Rd^2\varepsilon,$$

as required. □

8. Discretization

Lemma 5. *Let $\varepsilon, \rho \in \left(0, \frac{1}{2}\right)$. Let $K \in \mathcal{K}_R$. Let $x, x', y \in \partial K$ and let ν and ν' be the outer unit normals to ∂K at x and x' respectively. Assume that $|x - x'|^2 + |\nu - \nu'|^2 \leqslant \rho^2$ and $\langle y, \nu\rangle \geqslant \left(1 - \frac{\varepsilon}{2}\right)\langle x, \nu\rangle$. Then*

$$\langle y, \nu'\rangle \geqslant \left(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon d^2 + |y - x|)\right)\langle x', \nu'\rangle.$$

Proof. We have

$$\begin{aligned}\langle y, \nu'\rangle = \langle x, \nu'\rangle + \langle y - x, \nu'\rangle = \\ \langle x', \nu'\rangle + \langle x - x', \nu'\rangle + \langle y - x, \nu\rangle + \langle y - x, \nu' - \nu\rangle \geqslant \\ \langle x', \nu'\rangle + \langle x - x', \nu' - \nu\rangle + \langle y - x, \nu\rangle + \langle y - x, \nu' - \nu\rangle \geqslant \\ \langle x', \nu'\rangle - \rho^2 - \frac{\varepsilon}{2}\langle x, \nu\rangle - \rho|y - x|.\end{aligned}$$

Here, when passing from the second line to the third one, we used the inequality $\langle x - x', \nu\rangle \geqslant 0$.

Note now that

$$\langle x, \nu\rangle = \langle x, \nu'\rangle + \langle x, \nu - \nu'\rangle \leqslant \langle x', \nu'\rangle + \rho d^2$$

and $\langle x', \nu'\rangle \geqslant \frac{1}{2}$, so

$$\begin{aligned}\langle y, \nu'\rangle \geqslant \left(1 - \frac{\varepsilon}{2}\right)\langle x', \nu'\rangle - \rho\left(\rho + \frac{\varepsilon d^2}{2} + |y - x|\right) \geqslant \\ \left(1 - \frac{\varepsilon}{2} - 2\rho(\rho + \varepsilon d^2 + |y - x|)\right)\langle x', \nu'\rangle.\end{aligned}$$

□

Recall that our task is to find a finite set of points $Y \subset \partial K$ such that $(1 - \varepsilon)K \subset \operatorname{conv} Y$. This requirement is equivalent to the statement that for every $x \in \partial K$, there exists $y \in Y$ such that $\langle y, \nu\rangle \geqslant (1 - \varepsilon)\langle x, \nu\rangle$, where ν is the outer unit normal to ∂K at x.

Lemma 5 implies that it would suffice to show the existence of $y \in Y$ satisfying a slightly stronger inequality $\langle y, \nu\rangle \geqslant \left(1 - \frac{\varepsilon}{2}\right)\langle x, \nu\rangle$ for every point x in the Bronshtein–Ivanov net only, provided that we can ensure that $2\rho(\rho + \varepsilon d^2 + |y - x|) \leqslant \frac{\varepsilon}{2}$.

To this end, we apply Lemma 4, which shows that the inequality $\langle y, \nu \rangle \geqslant \left(1 - \frac{\varepsilon}{2}\right) \langle x, \nu \rangle$ automatically implies the distance bound $|y - x| \leqslant \sqrt{2R} d \sqrt{\frac{\varepsilon}{2}} = d\sqrt{R}\sqrt{\varepsilon}$. Thus, if we choose $\rho = \frac{1}{4(d^2+1+d\sqrt{R})}\sqrt{\varepsilon}$, we will be in good shape.

By Lemma 3, the size N of the corresponding Bronshtein-Ivanov net is at most $8^d(d^2+3)^d(d^2+1+d\sqrt{R})^d \varepsilon^{-\frac{d-1}{2}} = C(d)\varepsilon^{-\frac{d-1}{2}}$, which has the correct power of ε already. However, $C(d)$ is superexponential in d, which prevents us from just using the full Bronshtein–Ivanov net for Y and forces us to work a bit harder.

9. Rogers' trick

We now remind the reader a simple abstract construction essentially due to Rogers [R].

Lemma 6. *Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a family of measurable subsets of a probability space (U, μ) such that for some $\vartheta > 0$, we have $\mu(S_i) \geqslant \vartheta$ for all $i = 1, \ldots, N$. Then there exists a set Y of cardinality at most $\lceil \vartheta^{-1} \log(N\vartheta) \rceil + \vartheta^{-1}$ that intersects each S_i.*

Here $\lceil z \rceil$ stands for the least non-negative integer greater than or equal to z.

Proof. First we choose M points randomly and independently according to μ and obtain a random set Y_0. For every fixed $i \in \{1, \ldots, N\}$, we have

$$\mathbb{P}\{Y \cap S_i = \varnothing\} \leqslant (1-\vartheta)^M \leqslant e^{-\vartheta M}.$$

Hence, the expected number of sets $S_i \in \mathcal{S}$ disjoint from Y_0 is at most $Ne^{-\vartheta M}$. Choosing one additional point in each such set, we shall get a set Y of cardinality $Ne^{-\vartheta M} + M$ intersecting all S_i. Puting $M = \lceil \vartheta^{-1} \log(N\vartheta) \rceil$, we get the desired bound. □

Now, let $K \in \mathcal{K}_R$. Suppose that we can construct a probability measure μ on ∂K such that for every $x \in \partial K$ and every $\varepsilon > 0$, we have $\mu(S(x, \varepsilon)) \geqslant p\varepsilon^{\frac{d-1}{2}}$ with some $p > 0$.

We take the Bronshtein–Ivanov net X of K constructed in Section 6. Its cardinality N does not exceed $C(d)\varepsilon^{-\frac{d-1}{2}}$, where $C(d)$ is of order $e^{O(d \log d)}$. Consider the caps $S(x, \frac{\varepsilon}{2}), x \in X$. By Lemma 6, there exists a set $Y \subset \partial K$ of cardinality at most $\lceil 2^{\frac{d-1}{2}} p^{-1} \varepsilon^{-\frac{d-1}{2}} \log(C(d) 2^{-\frac{d-1}{2}} p) \rceil + 2^{\frac{d-1}{2}} p^{-1} \varepsilon^{-\frac{d-1}{2}}$ that intersects each of those caps. If $p = e^{O(d)}$, then the cardinality of Y is of order $e^{O(d)} \varepsilon^{-\frac{d-1}{2}}$.

10. The construction of the measure

Let n be a positive integer (we shall need both $n = d$ and $n = d-1$). Recall that for a convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, its polar body $K^\circ \subset \mathbb{R}^n$ is defined by

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \quad \text{for all} \quad x \in K\}.$$

We shall need the following well-known (but, in part, highly non-trivial) facts about the polar bodies:

Fact 1. If K has a smooth boundary and is strictly convex, that is, K contains no line segment on its boundary, then the relation $\langle x, x^*\rangle = 1$, $x \in \partial K$, $x^* \in \partial K^\circ$, defines a continuous one to one mapping * from ∂K to ∂K°. The vector x^* is just $\frac{\nu}{\langle x,\nu\rangle}$, where ν is the outer unit normal to ∂K at x (see [Sch], Corollary 1.7.3, page 40).

Fact 2. For any convex body $K \subset \mathbb{R}^n$ containing the origin in its interior, we have $\mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \geqslant e^{O(n)}n^{-n}$ (see [BM], [K], [NAZ]).

Fact 3. If K is a convex body with the center of mass at the origin, then

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K^\circ) \leqslant e^{O(n)}n^{-n}$$

(see [MP]).

Lemma 7. *Let $K \subset \mathbb{R}^d$ be a convex body containing the origin in its interior and satisfying the Santaló bound* $\mathrm{vol}_d(K)\,\mathrm{vol}_d(K^\circ) \leqslant e^{O(d)}d^{-d}$. *For any Borel set $S \subset \partial K$, define $S^* = \{x^* \in \partial K^\circ : x \in S\}$. Consider the "cones" $C(S) = \{rx : x \in S, 0 \leqslant r \leqslant 1\}$ and $C(S^*) = \{ry : y \in S^*, 0 \leqslant r \leqslant 1\}$ and put*

$$\mu(S) = \frac{1}{2}\Big(\frac{\mathrm{vol}_d(C(S))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K^\circ)}\Big).$$

Then μ is a probability measure on ∂K invariant under linear automorphisms of $\mathbb{R}^d$ and $\mu(S(x,\varepsilon)) \geqslant e^{O(d)}\varepsilon^{\frac{d-1}{2}}$ for all $x \in \partial K$ and all $\varepsilon \in (0, \frac{1}{2})$.

Proof. The invariance of μ under linear automorphisms of $\mathbb{R}^d$ follows immediately from the general properties of the volume with respect to linear transformations and the relation $(TK)^\circ = (T^{-1})^*K^\circ$.

Apply an appropriate linear transformation to put the body K in such a position that $x = x^* = e = (0, \dots, 0, 1) \in \mathbb{R}^d$. Then $S = S(x, \varepsilon)$ is given by $\langle x, e\rangle \geqslant 1 - \varepsilon$. Let $Q \subset e^\perp \cong \mathbb{R}^{d-1}$ be the convex body such that $(1-\varepsilon)e + Q$ is the cross-section of K by the hyperplane $\{x : \langle x, e\rangle = 1 - \varepsilon\}$. Let $\widetilde{K} = K \cap \{x : \langle x, e\rangle \leqslant 1 - \varepsilon\}$.

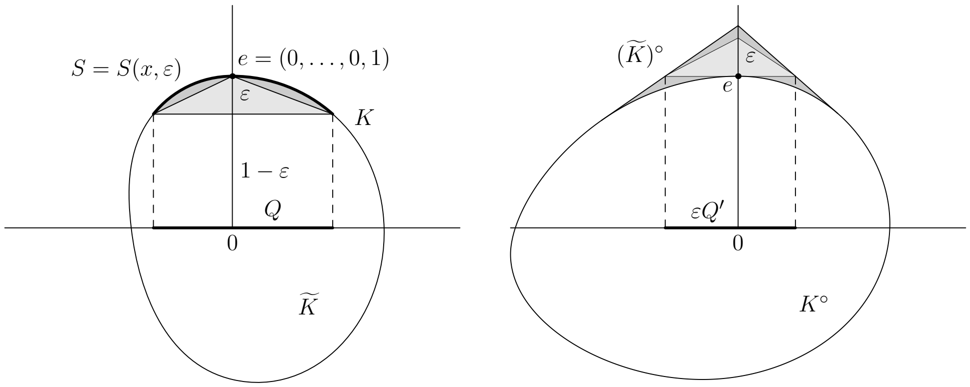

FIGURE 4. The regions $K \setminus \widetilde{K}$ and $(\widetilde{K})^\circ \setminus K^\circ$

Our first goal will be to show that

$$\mathrm{vol}_d(K \setminus \widetilde{K})\,\mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \geqslant \frac{1}{d^2}\varepsilon^{d+1}\,\mathrm{vol}_{d-1}(Q)\,\mathrm{vol}_{d-1}(Q'),$$

where $Q' \subset e^\perp$ is the polar body to Q in $\mathbb{R}^{d-1}$.

To this end, note that $K \setminus \widetilde{K}$ contains the interior of the pyramid $\mathrm{conv}(\{e\} \cup (1-\varepsilon)e + Q)$ of height ε with the base $(1-\varepsilon)e + Q$, so

$$\mathrm{vol}_d(K \setminus \widetilde{K}) \geqslant \frac{1}{d}\varepsilon\,\mathrm{vol}_{d-1}(Q).$$

We claim now that the interior of the pyramid $\Pi = \mathrm{conv}\{(1+\varepsilon)e, e+\varepsilon Q'\}$ is contained in $(\widetilde{K})^\circ \setminus K^\circ$ (see Figure 4). Since $K^\circ \subset \{y : \langle y, e\rangle \leqslant 1\}$, and $\mathrm{int}\,\Pi \subset \{y : \langle y, e\rangle > 1\}$, it suffices to show that $\Pi \subset (\widetilde{K})^\circ$.

To this end, take $x \in \widetilde{K}$, and let $\langle x, e\rangle = 1 - t\varepsilon$, $t \geqslant 1$, so $x = (1-t\varepsilon)e + x'$, where $x' \in e^\perp$.

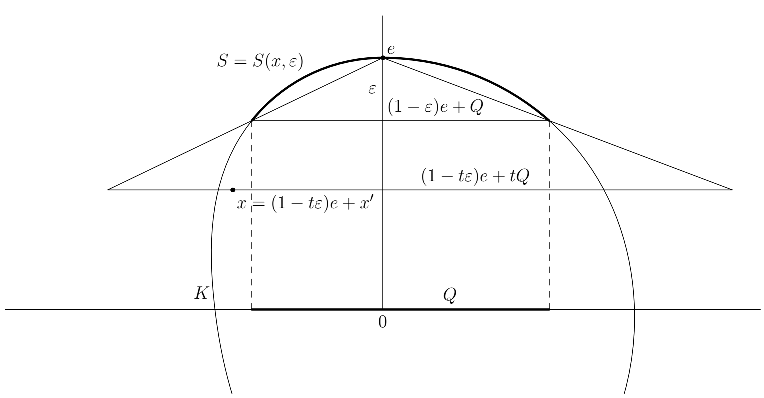

FIGURE 5. The cross-section of K by the hyperplane $\{x : \langle x, e\rangle = 1 - t\varepsilon\}$ is contained in tQ

Since $e \in K$, by the convexity of K, $x' \in tQ$ (see Figure 5). Now, $\langle x, (1+\varepsilon)e\rangle = (1-t\varepsilon)(1+\varepsilon) \leq 1$, hence, $(1+\varepsilon)e \in (\widetilde{K})^\circ$. Let $y = e+\varepsilon y'$ with $y' \in Q'$. Then $\langle x, y\rangle = 1 - t\varepsilon + \varepsilon\langle x', y'\rangle \leq 1 - t\varepsilon + t\varepsilon = 1$. Thus, $e + \varepsilon Q' \subset (\widetilde{K})^\circ$. It follows by the convexity of $(\widetilde{K})^\circ$ that $\Pi \subset (\widetilde{K})^\circ$, and, therefore,

$$\mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \geqslant \mathrm{vol}_d(\Pi) = \frac{1}{d}\varepsilon^d \,\mathrm{vol}_{d-1}(Q').$$

Multiplying these two estimates, we get the desired inequality.

On the other hand, we have $\mathrm{int}(K \setminus \widetilde{K}) \subset C(S) \setminus (1-\varepsilon)C(S)$, and $\mathrm{int}((\widetilde{K})^\circ \setminus K^\circ) \subset (1-\varepsilon)^{-1}C(S^*) \setminus C(S^*)$. Hence,

$$\begin{aligned}\mathrm{vol}_d(K \setminus \widetilde{K})\,\mathrm{vol}_d((\widetilde{K})^\circ \setminus K^\circ) \leqslant \\ (1-(1-\varepsilon)^d)((1-\varepsilon)^{-d} - 1)\,\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)) \leqslant \\ e^{O(d)}\varepsilon^2\,\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)).\end{aligned}$$

Combining it with the previous estimate and using Fact 2, we get

$$\begin{aligned}\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*)) \geqslant e^{O(d)}\varepsilon^{d-1}\,\mathrm{vol}_{d-1}(Q)\,\mathrm{vol}_{d-1}(Q') \geqslant \\ e^{O(d)}\varepsilon^{d-1}(d-1)^{-(d-1)}.\end{aligned}$$

Finally, since $\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ) \leqslant e^{O(d)}d^{-d}$, we get

$$\begin{aligned}\mu(S) \geqslant \frac{1}{2}\Big(\frac{\mathrm{vol}_d(C(S))}{\mathrm{vol}_d(K)} + \frac{\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K^\circ)}\Big) \geqslant \\ \sqrt{\frac{\mathrm{vol}_d(C(S))\,\mathrm{vol}_d(C(S^*))}{\mathrm{vol}_d(K)\mathrm{vol}_d(K^\circ)}} \geqslant e^{O(d)}\varepsilon^{\frac{d-1}{2}},\end{aligned}$$

as required. □

This lemma, together with the discussion in Section 9, completes the proof of the theorem.

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