NON-UNIQUENESS OF CONVEX BODIES WITH PRESCRIBED VOLUMES OF SECTIONS AND PROJECTIONS

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ABSTRACT. We show that if $d \geq 4$ is even, then one can find two essentially different convex bodies such that the volumes of their maximal sections, central sections, and projections coincide for all directions.

1. INTRODUCTION

As usual, a *convex body* $K \subset \mathbb{R}^d$ is a compact convex subset of $\mathbb{R}^d$ with non-empty interior. We assume that $0 \in K$. We consider the *central section function* A_K:

$$A_K(u) = \mathrm{vol}_{d-1}(K \cap u^{\perp}), \qquad u \in \mathbb{S}^{d-1}, \tag{1}$$

the *maximal section function* M_K:

$$M_K(u) = \max_{t \in \mathbb{R}} \mathrm{vol}_{d-1}(K \cap (u^{\perp} + tu)), \qquad u \in \mathbb{S}^{d-1}, \tag{2}$$

and the *projection function* P_K:

$$P_K(u) = \mathrm{vol}_{d-1}(K|u^{\perp}), \qquad u \in \mathbb{S}^{d-1}. \tag{3}$$

Here $u^{\perp}$ stands for the hyperplane passing through the origin and orthogonal to the unit vector u, $K \cap (u^{\perp} + tu)$ is the section of K by the affine hyperplane $u^{\perp} + tu$, and $K|u^{\perp}$ is the projection of K to $u^{\perp}$. Observe that $A_K \leq M_K \leq P_K$. It is well known, [Ga], that for origin-symmetric bodies *each* of the functions $M_K = A_K$ and P_K determines the convex body $K \subset \mathbb{R}^d$ uniquely. More precisely, either of the conditions

$$M_{K_1}(u) = M_{K_2}(u) \qquad \forall u \in \mathbb{S}^{d-1},$$

and

$$P_{K_1}(u) = P_{K_2}(u) \qquad \forall u \in \mathbb{S}^{d-1},$$

implies $K_1 = K_2$, provided K_1, K_2 are origin-symmetric and convex.

In this paper, we address the (im)possibility of analogous results for not necessarily symmetric convex bodies.

It is well known, [BF], that on the plane there are convex bodies K that are *not* Euclidean discs, but nevertheless satisfy $M_K(u) = P_K(u) = 1$ for all $u \in \mathbb{S}^1$. These are the *bodies of constant width* 1.

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In 1929 T. Bonnesen asked whether *every* convex body $K \subset \mathbb{R}^3$ is uniquely defined by P_K and M_K, (see [BF], page 51). We note that in any dimension $d \geq 3$, it is not even known whether the conditions $M_K \equiv c_1$, $P_K \equiv c_2$ are incompatible for $c_1 < c_2$.

In 1969 V. Klee asked whether the condition $M_{K_1} \equiv M_{K_2}$ implies $K_1 = K_2$, or, at least, whether the condition $M_K \equiv c$ implies that K is a Euclidean ball, see [Kl1].

Recently, R. Gardner and V. Yaskin, together with the second and the third named authors constructed two bodies of revolution K_1, K_2 such that K_1 is origin-symmetric, K_2 is not origin-symmetric, but $M_{K_1} \equiv M_{K_2}$, thus answering the first version of Klee's question but not the second one (see [GRYZ]).

The main results we will present in this paper are the following.

Theorem 1. *If $d = 4$, there exists a convex body of revolution $K \subset \mathbb{R}^d$ satisfying $M_K \equiv const$ that is not a Euclidean ball.*

Theorem 2. *If $d \geq 4$ is even, there exist two essentially different convex bodies of revolution K_1, $K_2 \subset \mathbb{R}^d$ such that $A_{K_1} \equiv A_{K_2}$, $M_{K_1} \equiv M_{K_2}$, and $P_{K_1} \equiv P_{K_2}$.*

Theorem 1 answers the question of Klee in $\mathbb{R}^4$, and Theorem 2 answers the analogue of the question of Bonnesen in even dimensions.

Remark 1. *Theorem 1 is actually true in all dimensions, but the construction for $d \neq 4$ is long and rather technical, so we will present it in a separate paper.*

We borrowed the general idea of the construction of the bodies K_1 and K_2 in Theorem 2 from [RY], which attributes it to [GV] and [GSW]. It can be easily understood from the following illustration.

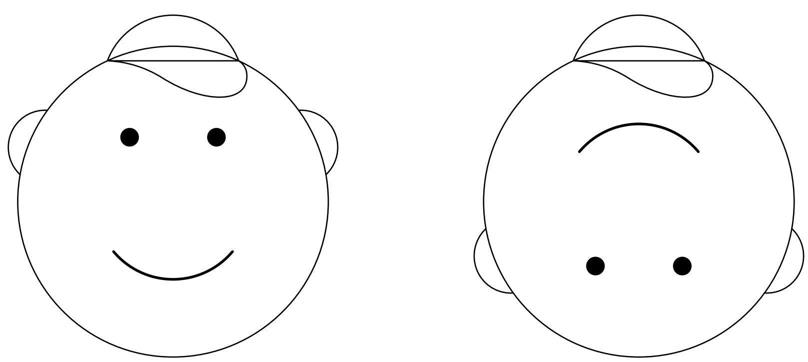

FIGURE 1. Two small-eared round faces in a cap

Here the "ears" and the "cap" will be made very small in order not to destroy the convexity of the bodies.

The paper is organized as follows. In Section 2 we reduce the problem to finding a non-trivial solution to two integral equations. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2.

2. Reduction to a system of integral equations

From now on, we assume that $d \geq 3$. We will be dealing with the bodies of revolution

$$K_f = \{x \in \mathbb{R}^d : x_2^2 + x_3^2 + ... + x_d^2 \leq f^2(x_1)\},$$

obtained by the rotation of a smooth (except for endpoints) concave function f supported on $[-1, 1]$ about the x_1-axis.

Note that K is rotation invariant, thus every its hyperplane section is equivalent to a section by a hyperplane with normal vector in the second quadrant of the (x_1, x_2)-plane.

Lemma 1. *Let $L(\xi) = L(s, h, \xi) = s\xi + h$ be a linear function with slope s, and let $H(L) = \{x \in \mathbb{R}^d : x_2 = L(x_1)\}$ be the corresponding hyperplane. Then the section $K \cap H(L)$ is of maximal volume if and only if*

$$\int_{-x}^{y} (f^2 - L^2)^{(d-4)/2} L = 0, \tag{4}$$

where $-x$ and y are the first coordinates of the points at which L intersects the graphs of $-f$ and f respectively (see Figure 2).

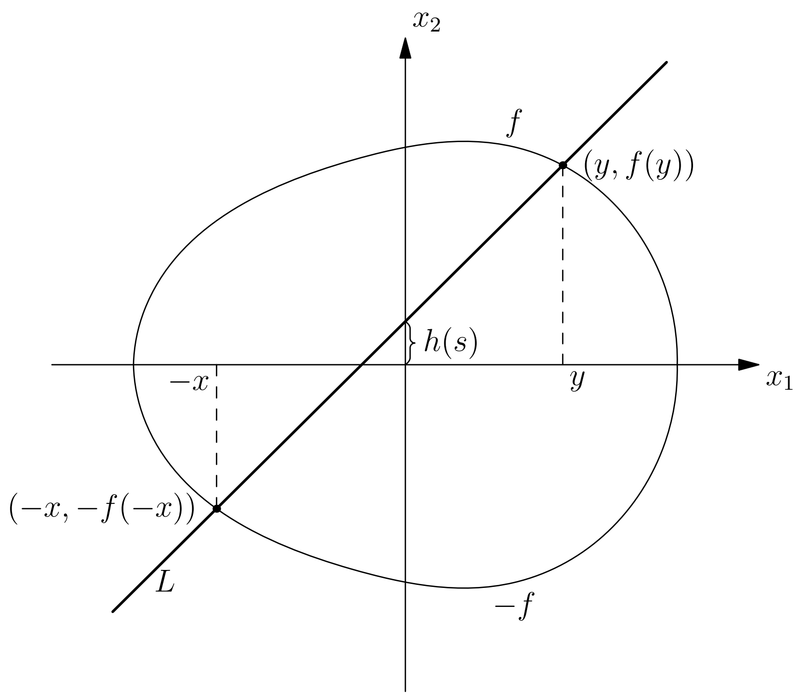

FIGURE 2. View of K and $H(L)$ in (x_1, x_2)-plane.

Proof. Fix $s > 0$. Observe that the slice $K \cap H(L) \cap H_\xi$ of $K \cap H(L)$ by the hyperplane $H_\xi = \{x \in \mathbb{R}^d : x_1 = \xi\}$, $-x(s) < \xi < y(s)$, is the $(d-2)$-dimensional Euclidean ball

$\{(\xi, L(\xi), x_3, x_4, ..., x_d) : x_3^2 + ... + x_d^2 \le r^2\}$ of radius $r = \sqrt{f^2(\xi) - L^2(\xi)}$. Hence,

$$\mathrm{vol}_{d-1}(K \cap H(L)) = v_{d-2}\sqrt{1+s^2} \int_{-x(s)}^{y(s)} (f^2(\xi) - L^2(\xi))^{(d-2)/2} d\xi, \tag{5}$$

where v_{d-2} is the volume of the unit ball in $\mathbb{R}^{d-2}$.

The section $K \cap H(L)$ is of maximal volume if and only if

$$\frac{d}{dh}\mathrm{vol}_{d-1}(K \cap H(L)) = 0,$$

where in the *only if* part we use the Theorem of Brunn, [Ga]. Computing the derivative, we conclude that for a given $s \in \mathbb{R}$, the section $K \cap H(L)$ is of maximal volume if and only if (4) holds. □

Lemma 2. *Let $L(s, \xi) = s\xi + h(s)$ be a family of linear functions parameterized by the slope s. For each L in our family, define the hyperplane $H(L)$ by $H(L) = \{x \in \mathbb{R}^d : x_2 = L(x_1)\}$, (see Figure 2). The corresponding family of sections is of constant $d-1$-dimensional volume if and only if*

$$\int_{-x}^{y} (f^2 - L^2)^{(d-2)/2} = \frac{const}{\sqrt{1+s^2}}, \qquad \textit{for all} \qquad s > 0. \tag{6}$$

Proof. The right hand side in (5) is constant if and only if (6) holds. □

3. The case $d = 4$

Observe that when $d = 4$, the system of equations (4), (6) simplifies to

$$\int_{-x}^{y} L = 0, \qquad \text{and} \qquad \int_{-x}^{y} (f^2 - L^2) = \frac{const}{\sqrt{1+s^2}}, \qquad \text{for all} \qquad s > 0.$$

In this case we will show that the *maximal sections correspond to level intervals*, see Proposition 1 below. We will also prove that *the values of the maximal section function M_K depend on the distribution function $t \to |\{f > t\}|$ only.* More precisely, we have

Theorem 3. *Let $d = 4$, $K = \{x \in \mathbb{R}^4 : x_2^2 + x_3^2 + x_4^2 \le f^2(x_1)\}$, and let*

$$u = u(s) = (-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}, 0, 0) \in \mathbb{S}^3, \qquad s > 0.$$

Then,

$$M_{K_f}(u) = \pi\sqrt{1+s^2}\Big(\frac{2}{3}\, t^2\, |\{f > t\}| + \int_t^{\infty} 2\tau|\{f > \tau\}| d\tau\Big), \tag{7}$$

where t is the unique solution of the equation $s = 2t/|\{f > t\}|$.

In particular, if f_1 and f_2 are equimeasurable (i.e., for every $\tau > 0$, we have $|\{f_1 > \tau\}| = |\{f_2 > \tau\}|$), then $M_{K_{f_1}} \equiv M_{K_{f_2}}$.

Theorem 3 is a simple consequence of the following two propositions.

Proposition 1. *Let $f, s, u(s)$ be as in Theorem 3. Then the section of maximal volume in the direction $u(s)$ is the one that corresponds to the line joining $(-x,-t)$ and (y,t), where t is such that $s = 2t/|\{f > t\}|$, $0 < t < \max_{\xi\in[-1,1]} f(\xi)$, (see Figure 3).*

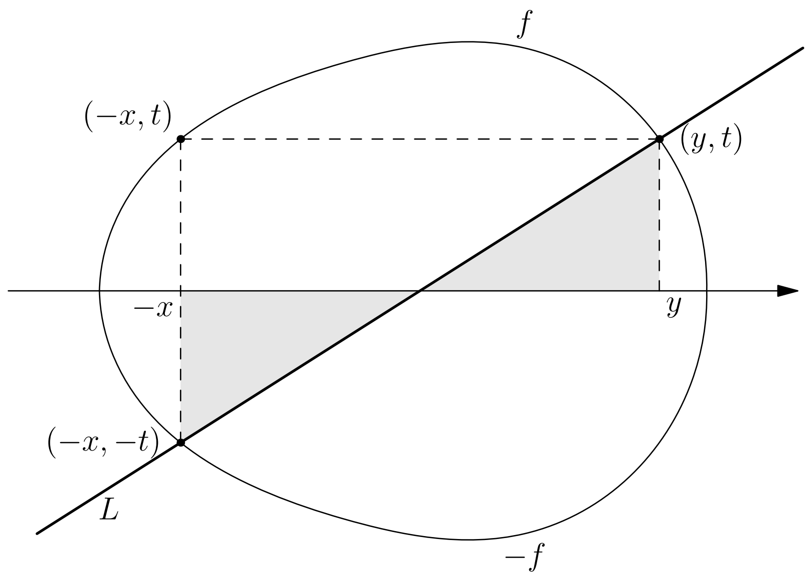

FIGURE 3. Maximal slice in $\mathbb{R}^4$

Proof. Fix $s > 0$. Since the distribution function is decreasing to 0, there exists a unique t satisfying $s = 2t/|\{f > t\}|$. To prove that

$$\int_{-x}^{y} L(\xi)d\xi = 0$$

observe that two shaded triangles on Figure 3 are congruent. □

Proposition 2. *Let K, f, t, x, y be as in the previous proposition, and let the line L be passing through the points $(-x,-t)$, (y,t). Then (7) holds.*

Proof. Note that

$$\int_{-x}^{y} L^2 = (x+y)\frac{t^2}{3} = \frac{t^2}{3}|\{f > t\}|,$$

and

$$\int_{-x}^{y} f^2 = \int_{\{f>t\}} f^2 = t^2|\{f > t\}| + \int_{t}^{\infty} 2\tau|\{f > \tau\}|d\tau.$$

□

Proof of Theorem 1. Let $f_o(\xi)=\sqrt{1-\xi^2}$, $\xi\in[-1,1]$. Take a concave function f on $[-1,1]$ such that $f\neq f_o$ and f is equimeasurable with f_o.

4. Proof of Theorem 2

Let φ and ψ be two smooth functions supported on the intervals $D=[\frac{1}{2}-\delta,\frac{1}{2}+\delta]$ and $E=[1-\delta,1]$ respectively, where $0<\delta<\frac{1}{8}$. Define

$$f_+(\xi)=f_o(\xi)+\varepsilon\varphi(\xi)-\varepsilon\varphi(-\xi)+\varepsilon\psi(\xi),$$

and

$$f_-(\xi)=f_o(\xi)-\varepsilon\varphi(\xi)+\varepsilon\varphi(-\xi)+\varepsilon\psi(\xi),$$

where $\varepsilon>0$ is so small that $f_\pm$ are concave on $[-1,1]$ (see Figure 4).

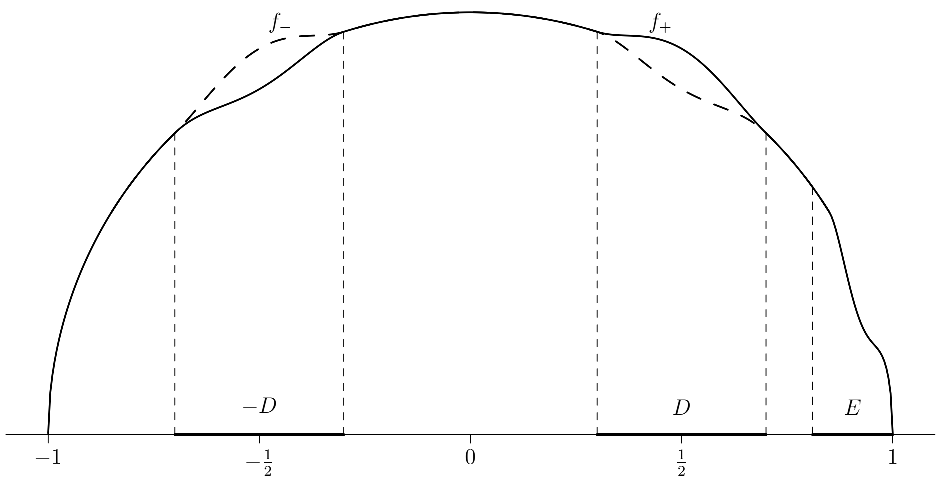

Figure 4. Graph of functions $f_\pm$

Define $K_1=K_{f_+}$ and $K_2=K_{f_-}$.
Observe that

(8) $f_+(\xi)=f_-(\xi)\ \forall\xi\in[-1,1]\setminus(D\cup(-D)),$ and $f_+(\xi)=f_-(-\xi)\ \forall\xi\in D\cup(-D).$

We can choose ε so small that K_1 and K_2 are very close to the Euclidean ball, and the sections of the maximal volume of K_1 and K_2 are very close to the central sections of the ball.

In particular, the intersection points $x=x(s)$ and $y=y(s)$ satisfy

(9) $$x(s),\,y(s)>\frac{5}{8}\ \text{ if }\ s\le\frac{\sqrt{7}}{3},\qquad\text{and}\qquad x(s),\,y(s)<\frac{7}{8}\ \text{ if }\ s\ge\frac{\sqrt{7}}{3},$$

for both bodies.

First, we show that $P_{K_1}\equiv P_{K_2}$ and $A_{K_1}\equiv A_{K_2}$.

Observe that we have $h_{K_1}(u)=h_{K_2}(u)$ and $\rho_{K_1}(u)=\rho_{K_2}(u)$ for all directions $u=(\xi,\sqrt{1-\xi^2},0,0,...,0)\in\mathbb{S}^{d-1}$, $\xi\in[0,1]\setminus D$. Observe also that $h_{K_1}(u)=h_{K_2}(-u)$

and $\rho_{K_1}(u) = \rho_{K_2}(-u)$ for all directions $u = (\xi, \sqrt{1-\xi^2}, 0, 0, ..., 0) \in \mathbb{S}^{d-1}$, $\xi \in D$. Hence, the non-ordered pairs

$$\{h_{K_1}(u), h_{K_1}(-u)\}, \qquad \text{and} \qquad \{h_{K_2}(u), h_{K_2}(-u)\}$$

coincide for all $u \in \mathbb{S}^{d-1}$, and so do the pairs

$$\{\rho_{K_1}(u), \rho_{K_1}(-u)\}, \qquad \text{and} \qquad \{\rho_{K_2}(u), \rho_{K_2}(-u)\}.$$

By the result of Goodey, Schneider and Weil, [GSW], we have $P_{K_1} \equiv P_{K_2}$. Also,

$$A_{K_1}(\theta) = \frac{1}{d-1} \int\limits_{\mathbb{S}^{d-1}\cap\theta^{\perp}} \frac{\rho_{K_1}^{d-1}(-u) + \rho_{K_1}^{d-1}(u)}{2} d\sigma(u) =$$

$$\frac{1}{d-1} \int\limits_{\mathbb{S}^{d-1}\cap\theta^{\perp}} \frac{\rho_{K_2}^{d-1}(-u) + \rho_{K_2}^{d-1}(u)}{2} d\sigma(u) = A_{K_2}(\theta)$$

for all $\theta \in \mathbb{S}^{d-1}$.

It remains to show that

$$M_{K_1} \equiv M_{K_2}. \tag{10}$$

Assume first that $d = 4$. The functions f_+ and f_- are equimeasurable, so (10) follows from Theorem 3.

Let now $d \geq 6$ be even. Note that in this case, $p = \frac{d-2}{2} \in \mathbb{N}$.

We claim that we can choose φ such that (10) holds. This result will be a consequence of the following two propositions.

Proposition 3. *If ε is small enough, then for*

$$u = u(s) = \left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}, 0, 0, ..., 0\right) \in \mathbb{S}^{d-1}$$

we have

$$M_{K_1}(u) = M_{K_2}(u), \tag{11}$$

provided $s \geq \frac{\sqrt{7}}{3}$.

Proof. By (9), if the sections $K_1 \cap H(L_1)$, $K_2 \cap H(L_2)$ are the sections of the maximal volume, corresponding to the same slope $s \geq \frac{\sqrt{7}}{3}$, then, they are the sections of two symmetric (to each other) bodies corresponding to $\psi = 0$. Hence, (11) holds by symmetry. □

To formulate the second proposition we will need the following result, which is a consequence of the Borsuk-Ulam Theorem.

Lemma 3. *There exists $\varphi \neq 0$ such that*

$$\int\limits_{-\frac{5}{8}}^{\frac{5}{8}} f_+^{2j}(\xi)\xi^l d\xi = \int\limits_{-\frac{5}{8}}^{\frac{5}{8}} f_-^{2j}(\xi)\xi^l d\xi \tag{12}$$

for all $j = 0, ..., p$ and $l = 0, ..., 2(p-j)$.

Proof. We will choose φ using the Borsuk-Ulam Theorem.

For j and l as above, consider the vector $\mathbf{a} = \mathbf{a}(\varphi)$ with coordinates

$$a_{j,l}(\varphi) = \int_{-\frac{5}{8}}^{\frac{5}{8}} f_+^{2j}(\xi)\xi^l d\xi - \int_{-\frac{5}{8}}^{\frac{5}{8}} f_-^{2j}(\xi)\xi^l d\xi.$$

We will view this vector as an element of $\mathbb{R}^{n(p)}$ with appropriately chosen $n(p)$.

For $x = (x_0, x_1, ..., x_{n(p)}) \in \mathbb{S}^{n(p)}$ define $\varphi_x = \sum_{j=0}^{n(p)} x_j\varphi_j$, where φ_j are smooth not identically zero functions with pairwise disjoint supports contained in D, and let

$$\mathbf{B} : \ (x_0, x_1, ..., x_{n(p)}) \ \rightarrow \ \varphi_x \ \rightarrow \ \mathbf{a}(\varphi_x)$$

be our map from $\mathbb{S}^{n(p)}$ to $\mathbb{R}^{n(p)}$. By the definition of $f_\pm$ the map $\mathbf{B}$ is *odd*. Hence, by the Borsuk-Ulam Theorem, one can choose x (and hence φ_x) in such a way that $\mathbf{a}(\varphi_x) = \mathbf{0}$. □

Proposition 4. *Let $L = L(s, h, \xi)$ be as above and let $0 \le s \le \frac{\sqrt{7}}{3}$. Then,*

$$\int_{-x}^{y} (f_-^2 - L^2)^p = \int_{-x}^{y} (f_+^2 - L^2)^p. \tag{13}$$

Moreover,

$$\int_{-x}^{y} (f_+^2 - L^2)^{p-1} L = 0 \qquad \text{if and only if} \qquad \int_{-x}^{y} (f_-^2 - L^2)^{p-1} L = 0. \tag{14}$$

In particular, (11) holds for $0 \le s \le \frac{\sqrt{7}}{3}$.

Proof. We start with the proof of (13). We open parentheses and observe that all we need to prove is

$$\int_{-x(s)}^{y(s)} f_+^{2j}(\xi)\xi^l d\xi = \int_{-x(s)}^{y(s)} f_-^{2j}(\xi)\xi^l d\xi$$

for all $0 \le s \le \frac{\sqrt{7}}{3}$, and for all $j = 0, ..., p$ and $l = 0, ..., 2(p - j)$. By (9), this follows from (12) since $f_+(\xi) = f_-(\xi)$ for $|\xi| \ge \frac{5}{8}$.

Similarly, (12) implies (14) for $j = 0, ..., p-1$ and $l = 0, ..., 2(p-1-j)+1$. □

Thus, (10) follows from Propositions 3, 4. This finishes the proof of the Theorem.

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