HARDY SPACES AND PARTIAL DERIVATIVES OF CONJUGATE HARMONIC FUNCTIONS

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Abstract. In this paper we give necessary and sufficient conditions for a harmonic vector and all its partial derivatives to belong to $H^p(\mathbf{R}_+^{n+1})$ for all $p > 0$. We also obtain the Hardy-Littlewood-Sobolev imbedding-type result, formulated on the language of the conjugate harmonic functions.

1. Introduction and statements of main results

Let $f(z)$ be an analytic function in the unit disc $\mathbf{D} = \{z : |z| < 1\}$ and let

$$M_p(r, f) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}, \qquad p > 0, \qquad 0 \leq r < 1. \tag{1}$$

It is well-known (see, for example, [17], [8], and references therein) that $f \in H^p(\mathbf{D})$, $p > 0$, if $M_p(r, f) \leq C < \infty$. It is also well-known that $f \in H^p(\mathbf{D})$ if and only if a subharmonic function $|f(z)|^p$ has a harmonic majorant. Using the Riemann conformal mapping theorem one can define the Hardy space on any domain $\mathbf{U}$ having more than one point in its boundary. An analytic function in a domain $\mathbf{U}$ belongs to $H^p(\mathbf{U})$, $p > 0$, if and only if a subharmonic function $|f(z)|^p$ has a harmonic majorant in $\mathbf{U}$. In particular, this is true in the case of the upper half-plane domain $\mathbf{U} = \mathbf{R}_+^2 = \{z : Im\, z > 0\}$. On the other hand, if we want to define $H^p(\mathbf{R}_+^2)$ in terms of the integral everages of type (1), one has to take into account a weight appearing after a change of variables in the integral. This is one of the reasons for an appearence of another natural class of functions, $h^p(\mathbf{R}_+^2)$, [6].

An analytic function $f \in h^p(\mathbf{R}_+^2)$, $p > 0$, if

$$M_p(y, f) = \Big(\int_{-\infty}^{\infty} |f(x + iy)|^p dx\Big)^{1/p} \leq C < \infty.$$

It is clear that not every bounded analytic function in the upper half-plane belongs to $h^p(\mathbf{R}_+^2)$. Moreover, the above condition is only sufficient for the existence of a harmonic majorant for $|f(x + iy)|^p$, and analytic functions from $h^p(\mathbf{R}_+^2)$ loose several important properties that analytic functions from $H^p(\mathbf{D})$ have. For example, if $f \in H^p(\mathbf{D})$, then $f \in H^q(\mathbf{D})$ for all $0 < q < p$. For $f \in h^p(\mathbf{R}_+^2)$, the statement $f \in H^q(\mathbf{R}_+^2)$, $0 < q < p$, is no longer true without additional assumptions on f.

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The reason for this is very simple, f is bounded in $|z| \leq r_0$, $0 < r_0 < 1$, and $f \in H^q(|z| \leq r_0)$, for all $q > 0$, since $M_p(r, f) \leq C(r_0)$, $0 < r \leq r_0$. This is why the class $S^p(\mathbf{R}^2_+)$, $p > 0$, comes into play, [7], [1].

An analytic function $f \in S^p(\mathbf{R}^2_+)$, $p > 0$, if for any $y_0 > 0$ there exists a constant $C(y_0, f)$, such that $M_p(y, f) \leq C(y_0)$ $\forall y \geq y_0$. In particular, if C is independent of y_0, then $f(x+iy) \in h^p(\mathbf{R}^2_+)$. Now, if $f \in S^p(\mathbf{R}^2_+)$, $p > 0$, and $y \to \infty$, then $f(x+iy)$ converges to zero uniformly with respect to x, and $f \in S^q(\mathbf{R}^2_+)$ $\forall q \geq p$. But again, the condition $f \in S^p(\mathbf{R}^2_+)$ does not imply $f \in S^r(\mathbf{R}^2_+)$, $0 < r \leq p$. Consider, for example, $f(x+iy) = u(x, y) + iv(x, y)$, where

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = \frac{y}{x^2 + y^2}.$$

We have $f \in S^p(\mathbf{R}^2_+)$, $p > 1$, but $f \notin S^q(\mathbf{R}^2_+)$, $0 < q \leq 1$.

Thus, in the case of the upper half-plane, one can consider different problems concerning classes $H^p(\mathbf{R}^2_+)$, $h^p(\mathbf{R}^2_+)$, $S^p(\mathbf{R}^2_+)$. In particular, we have $h^p(\mathbf{R}^2_+) \subset S^p(\mathbf{R}^2_+)$, and $h^p(\mathbf{R}^2_+) \subset H^p(\mathbf{R}^2_+)$.

In the case of the half-space, $\mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times (0, \infty)$, classes $h^p(\mathbf{R}^{n+1}_+)$, $S^p(\mathbf{R}^{n+1}_+)$, (all definitions are given in Section 2), were considered by Solomentsev [11], and by Stein and Weiss, [13], [15]. One of the results, proved in [13] reads as follows: if $F = (U(x, y), V_1(x, y), V_2(x, y), ..., V_n(x, y))$ is a harmonic vector, $(x, y) \in \mathbf{R}^{n+1}_+$, then $|F|^p$ is subharmonic, provided $p \geq (n-1)/n$. This leads to additional technical difficulties for $p < (n-1)/n$.

Finally, Fefferman and Stein [4], introduced the classes of gradients $H^p(\mathbf{R}^{n+1}_+)$, $p > 0$, of $\nabla^k F$, and showed that $|\nabla^k F|^p$ is subharmonic provided $p \geq (n-1)/(n+k-1)$, $k \in \mathbf{N}$. They also proved that $H^p(\mathbf{R}^{n+1}_+) = h^p(\mathbf{R}^{n+1}_+)$ for $p \geq (n-1)/n$. Nevertheless, Wolff [16] showed that $H^p(\mathbf{R}^{n+1}_+) \subsetneq h^p(\mathbf{R}^{n+1}_+)$ for $p < (n-1)/n$.

In this article we find new relations between the conjugate harmonic functions in $\mathbf{R}^{n+1}_+$ and their partial derivatives. We study the following problem: what can we say about conjugate harmonic functions, provided we are given certain restrictions, imposed on partial derivatives of a harmonic vector $F = (U, V_1, V_2, ..., V_n)$. We give necessary and sufficient conditions for a harmonic vector and all its partial derivatives up to the order k to belong to $H^p(\mathbf{R}^{n+1}_+)$, $p > 0$. Our first results are

Theorem 1. *Let $k \in \mathbf{N}$, $p > 0$. The harmonic vector $F = (U, V_1, ..., V_n)$ and all its partial derivatives of the order $\leq k$ belong to H^p if and only if*

$$(2) \qquad 1)\ M_p(y+1, F) \leq C, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \geq y} |D^k_{n+1} U(x, \eta)|\Big)^p dx \leq C.$$

Theorem 2. *Let $0 < p < q$. The harmonic vector $F = (U, V_1, ..., V_n)$ and all its partial derivatives of the order $\leq k$ belong to H^r, $p \leq r \leq q$, if and only if*

$$(3) \qquad 1)\ M_p(y+1, F) \leq C, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \geq y} |D^k_{n+1} U(x, \eta)|\Big)^q dx \leq C.$$

Conditions (2) allow to obtain the Hardy-Littlewood-Sobolev imbedding-type result, formulated on the language of the conjugate harmonic functions. Results of this type were obtained by Fefferman-Stein, [4], Flett [5], and others (see [12] and references therein). For us the case of small $p > 0$ is of special interest.

Theorem 3. *Let $k \in \mathbf{N}$, $p > 0$, and let F be as in Theorem 1.*

a) If $0 < kp < n$, then $F \in H^r$ $\forall r : p \le r \le np/(n - kp)$. If $kp \ge n$, then $F \in H^r$ $\forall r \ge p$.

b) Let $k > 1$, $1 \le m \le k - 1$, $m \in \mathbf{N}$. If $0 < (k - m)p < n$, then all partial derivatives of F of the order m belong to H^r $\forall r : p \le r \le np/(n - (k - m)p)$. If $(k - m)p \ge n$, then all partial derivatives of F of the order m belong to H^r $\forall r \ge p$.

One of the methods of the proof of Theorems 1 and 2 is the application of classes $S^p(\mathbf{R}^{n+1}_+)$ together with the Lagrange mean-value Theorem. We note that the first condition in (2) is natural not only because of the decomposition of the function into two parts, "$S^p(\mathbf{R}^{n+1}_+)$" and "$H^p(\mathbf{R}^{n+1}_+)$" (see Section 5). In fact, it (together with the second condition) implies $M_p(y + y_0, F) \le C(y_0)$ $\forall y_0 > 0$. Moreover, the next result shows that it is necessary.

Theorem 4. *Let $p > 0$, and let F be a harmonic vector such that*

$$1)\, F \rightrightarrows^{x}_{y\to\infty} 0, \qquad 2)\, M_p(y, D_{n+1}U) \le C, \qquad 3)\, |D_{n+1}U(x, y)| \le C.$$

Then

a) $F \in H^r$, $r > np/(n - p)$, provided $0 < p < n$.

b) If $p \ge n$, then there exist conjugate harmonic functions such that $F \notin H^r$, $r > 0$.

The paper is organized as follows. In section 2 we give all necessary definitions and auxiliary results used in the sequel. Section 3 is devoted to results needed for the proof of Theorem 1, and in section 4 we prove Theorem 1. In sections 5, 6 we prove Theorem 2. This result is used as a tool for the proof of Theorem 3. Auxiliary results for the proof of Theorem 3 are given in section 7. We prove Theorem 3 in section 8. The last section is devoted to the proof of Theorem 4. For convenience of the reader we split our proofs into elementary Lemmata.

2. Auxiliary results

We begin with the definition of $h^p(\mathbf{R}^{n+1}_+)$.

Let $U(x, y)$ be a harmonic function in $\mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times (0, \infty)$. We say that the vector-function $V(x, y) = (V_1(x, y), ..., V_n(x, y))$ is the conjugate of $U(x, y)$ in the sense of M. Riesz [12], [14], if $V_k(x, y)$, $k = 1, ..., n$ are harmonic functions, satisfying the generalized Cauchy-Riemann conditions:

$$\frac{\partial U}{\partial y} + \sum_{k=1}^{n} \frac{\partial V_k}{\partial x_k} = 0, \qquad \frac{\partial V_i}{\partial x_k} = \frac{\partial V_k}{\partial x_i}, \qquad \frac{\partial U}{\partial x_i} = \frac{\partial V_i}{\partial y}, \qquad i \ne k,\ k = 1, ..., n.$$

If $U(x, y)$ and $V(x, y)$ are conjugate in $\mathbf{R}^{n+1}_+$ in the above sense, then the vector-function

$$F(x, y) = (U(x, y), V(x, y)) = (U(x, y), V_1(x, y), ..., V_n(x, y))$$

is called a harmonic vector.

Define

$$M_p(y) = M_p(y, F) = \Big\{ \int\limits_{\mathbf{R}^n} |F(x,y)|^p dx \Big\}^{1/p}, \qquad p > 0.$$

Definition 1 ([1], [7]). *We say that $F(x, y) \in S^p(\mathbf{R}^{n+1}_+)$, $p > 0$ if for any $y_0 > 0$ there exists a constant $C(y_0, F)$, such that $\forall y \geq y_0$, $M_p(y, F) \leq C(y_0)$. In particular, if C is independent of y_0, then $F(x, y) \in h^p(\mathbf{R}^{n+1}_+)$.*

Now we define the space $H^p(\mathbf{R}^{n+1}_+)$. We follow the work of Fefferman and Stein[4]. Let $U(x, y)$ be a harmonic function in $\mathbf{R}^{n+1}_+$, and let $U_{j_1 j_2 j_3 \cdots j_k}$ denote a component of a symmetric tensor of rank k, $0 \leq j_i \leq n$, $i = 1, ..., n$. Suppose also that the trace of our tensor is zero, meaning

$$\sum_{j=0}^{n} U_{jjj_3 \dots j_k}(x, y) = 0, \qquad \forall j_3, ..., j_k.$$

The tensor of rank $k+1$ can be obtained from the above tensor of rank k by passing to its gradient:

$$U_{j_1 j_2 \dots j_k j_{k+1}}(x, y) = \frac{\partial}{\partial x_{j_{k+1}}} (U_{j_1 j_2 j_3 \dots j_k}(x, y)), \qquad x_0 = y, \; 0 \leq j_{k+1} \leq n.$$

Definition 2 ([4]). *We say that $U(x, y) \in H^p(\mathbf{R}^{n+1}_+)$, $p > 0$, if there exists a tensor of rank k of the above type with the properties:*

$$U_{0\dots0}(x, y) = U(x, y), \qquad \sup_{y>0} \int\limits_{\mathbf{R}^n} \Big(\sum_{(j)} U^2_{(j)}(x, y) \Big)^{p/2} dx < \infty, \qquad (j) = (j_1, \dots j_k).$$

It is well-known that the function $\Big(\sum\limits_{(j)} U^2_{(j)}(x, y) \Big)^{p/2}$ is subharmonic for $p \geq p_k = (n-k)/(n+k-1)$, see [3],[4],[14].

We will use the "radial" and nontangential maximal functions:

$$F^+(x) = \sup_{y>0} |F(x, y)|, \qquad N_\alpha(F)(x^0) = \sup_{(x,y)\in\Gamma_\alpha(x^0)} |F(x, y)|.$$

Here

$$\Gamma_\alpha(x^0) = \{(x, y) \in \mathbf{R}^{n+1}_+ : \, |x - x^0| < \alpha y\}, \qquad \alpha > 0,$$

is an infinite cone with the vertex at x^0. It is well-known [4] that

$$F(x, y) \in H^p(\mathbf{R}^{n+1}_+) \iff N_\alpha(F)(x) \in L^p \iff F^+(x) \in L^p, \; p > 0.$$

We also define the *weak maximal function*

$$WF(x, y) = \sup_{\zeta \geq y} |F(x, \zeta)|, \qquad y > 0.$$

The above expression is understood as folows: we fix x, and for fixed y we find the supremum over all $\zeta \geq y$.

We will repeatedly use the following results.

Lemma 1. *([4], p.173). Suppose $u(x,y)$ is harmonic in $\mathbf{R}^{n+1}_+$, and for some p, $0<p<\infty$,*

$$\sup_{y>0}\int\limits_{\mathbf{R}^n}|u(x,y)|^p dx<\infty,$$

then

$$\sup_{x\in\mathbf{R}^n}|u(x,y)|\le Ay^{-n/p},\qquad 0<y<\infty. \tag{4}$$

Theorem 5. *([5], p. 268). Let $0<p\le 1$, $k\in\mathbf{N}$, and let $u:\mathbf{R}^{n+1}_+\to\mathbf{R}$ be a harmonic function such that*

$$u(x,t)\rightrightarrows^{x}_{y\to\infty}0,\qquad K_{k,p}\equiv\int\limits_{\mathbf{R}^{n+1}_+}t^{kp-1}|D^k_{n+1}u(x,t)|^p dxdt<C.$$

Then $u(x,0)=\lim\limits_{t\to 0+}u(x,t)$ exists for almost all $x\in\mathbf{R}^n$, and for all $t\ge 0$,

$$\int\limits_{\mathbf{R}^n}|u(x,t)|^p dx\le AC(k,n,p)K_{k,p}.$$

Theorem 6. *([5], p. 269). Let $m\in\mathbf{N}$, $p\ge(n-1)/(m+n-1)$ (if $n=1$ we suppose $p>0$), and let $u:\mathbf{R}^{n+1}_+\to\mathbf{R}$ be harmonic. Then, for all $t>0$,*

$$\int\limits_{\mathbf{R}^n}|\nabla^m u(x,t)|^p dx\le A(m,n,p)t^{-mp-1}\int\limits_{t/2}^{3t/2}ds\int\limits_{\mathbf{R}^n}|u(x,t)|^p dx.$$

Corollary 1. *([5], p. 270). Let p,m be as in Theorem 6, let $b>0$, and let $u:\mathbf{R}^{n+1}_+\to\mathbf{R}$ be a harmonic function such that for all $t>0$*

$$\int\limits_{\mathbf{R}^n}|u(x,t)|^p dx\le Ct^{-b}.$$

Then

$$\int\limits_{\mathbf{R}^n}|\nabla^m u(x,t)|^p dx\le A(b,m,n,p)Ct^{-b-mp},\qquad (t>0).$$

In fact, the choice of p in Theorem 5 and Corollary 1 may be independent of m (see Lemma 2).

Theorem 7. [10]. *Let $p>0$ and let $F(x,y)=(U,V_1,...,V_n)$ be a harmonic vector satisfying*

$$1)\ V_i(x,y)\rightrightarrows^{x}_{y\to\infty}0,\ i=1,...,n,\qquad 2)\ M_p(y,U)\le C,\qquad 3)\ |U|\le C. \tag{5}$$

Then $F\in H^r$, $r>p$.

Notation. We denote by $D_i^k f(x,y)$ the partial derivative of the function f of the order k with respect to x_i, $i = 1,2,...,n+1$. $M(f)(x)$ denotes the usual Hardy-Littlewood maximal function of $f(x)$. The notation $f(x,y) \rightrightarrows^{x}_{y\to\infty} 0$ means that $f(x,y)$ converges to 0 uniformly with respect to x, provided $y \to \infty$, $\nabla^k f(x) = (\frac{\partial^k f(x)}{\partial x_1^k}, ..., \frac{\partial^k f(x)}{\partial x_n^k})$. Everywhere below the constants $A(k,n), C, K$ depend only on the parameters pointed in parentheses, and may be different from time to time.

3. Auxiliary lemmata for the proof of Theorem 1.

The next result shows that in Theorem 6 and Corollary 1 the choice of $p > 0$ may be independent on $m \in \mathbf{N}$. We include it here for convenience of the reader.

Lemma 2. *Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be such that $V_i \rightrightarrows^{x}_{y\to\infty} 0$, $i = 1,...,n$, $M_p(y,U) \le C$. Then*

$$M_p(y, \nabla^k F) \le ACy^{-k}, \qquad k \in \mathbf{N}.$$

Proof. By induction on k. Let $k = 1$. Fix $p > 0$ and let $l = \inf\{j \in \mathbf{N} : p \ge (n-1)/(j+n-1)\}$. Let $\phi_{ij}(x,y)$ be a coordinate of $\nabla V_i(x,y)$, $j = 1,...,n+1$, $x_{n+1} = y$, $i = 0,...,n$, $V_0 = U$. Since $\nabla V_i(x,y) \rightrightarrows^{x}_{y\to\infty} 0$, we may use the following relation (see [5] or [4])

$$\phi_{ij}(x,y) = \frac{1}{(2l-2)!}\int_y^\infty (s-y)^{2l-2} D_{n+1}^{2l-1}\phi_{ij}(x,s)ds =$$

$$\frac{1}{(2l-2)!}\int_0^\infty s^{2l-2} D_{n+1}^{2l-1}\phi_{ij}(x,s+y)ds.$$

We have $|\phi_{ij}(x,y)| \le h_{ij}(x,y)$, where

$$h_{ij}(x,y) \equiv \frac{1}{(2l-2)!}\int_0^\infty s^{2l-2}|\nabla^l D_{n+1}^{l-1}\phi_{ij}(x,s+y)|^p ds.$$

Theorem 3 of[5] implies (take $w = \nabla^l D_{n+1}^{l-1}\phi_{ij}$, $a = 2l-1$, $A = A(l,n,p)$),

$$\int_{\mathbf{R}^n} |\phi_{ij}(x,y)|^p dx \le \int_{\mathbf{R}^n} |h_{ij}(x,y)|^p dx \le A\int_0^\infty s^{(2l-1)p-1} ds \int_{\mathbf{R}^n} |\nabla^l D_{n+1}^{l-1}\phi_{ij}(x,s+y)|^p dx.$$

Since $D_{n+1}^{l-1}\phi_{ij}(x,y)$ is the $l-$th derivative of V_i, we use Theorem 6 to get

$$\int_{\mathbf{R}^n} |\nabla^l D_{n+1}^{l-1}\phi_{ij}(x,y)|^p dx \le \int_{\mathbf{R}^n} |\nabla^{2l} F(x,y)|^p dx \le C\, y^{-2lp}. \tag{6}$$

This gives

$$\int_{\mathbf{R}^n} |\phi_{ij}(x,y)|^p dx \le A(l,n,p)C\int_0^\infty s^{(2l-1)p-1}(s+y)^{-2lp} ds = A(l,n,p)Cy^{-p},$$

and the first induction step is proved.

Assume that the statement is true for $k-1$. Then $M_p(y, \nabla^{k-1}F) \le ACy^{-(k-1)}$. To prove it for k we define l as above and apply Corollary 1 with $b = k-1$, $m = 1$, $u = \nabla^{k-1}F$. □

Lemma 3. *Let $D_iU \rightrightarrows^{x}_{y\to\infty} 0$, $i = 1, ..., n$, and let*

$$M_p(y, D_{n+1}U) \le Cy^{-1} \tag{7}$$

for some $p > 0$. Then

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}U(x, y + y_0)|\Big)^p dx \le AC, \qquad \forall y_0 > 0,\ A = A(n, p, y_0). \tag{8}$$

Proof. Let $p > 1$. Then (see [12])

$$\|\sup_{y>0} |D_{n+1}U(\cdot, y + y_0)|\|_p \le CM_p(y + y_0, D_{n+1}U) \le AC, \qquad A = A(n, p, y_0).$$

Now let $0 < p \le 1$. Assume that

$$\int\limits_0^\infty \int\limits_{\mathbf{R}^n} s^{p-1}|\nabla^2U(x, s + y_0)|^p dxds \equiv \int\limits_0^1 \int\limits_{\mathbf{R}^n} + \int\limits_1^\infty \int\limits_{\mathbf{R}^n} < C(y_0) < \infty. \tag{9}$$

Applying Lemma 2, Theorem 5 (with $D_{n+1}U$ instead of u and $k = 1$) and the tensor representation of $D_{n+1}U$ from [4] we have $D_{n+1}U(x, y + y_0) \in H^p$. Thus, we prove (9). By Theorem 6 , (7) yields

$$M_p(y, \nabla^2U) \le ACy^{-2}. \tag{10}$$

This gives (9). Indeed, the first integral in the right-hand side of (9) is finite, since

$$M_p(y + y_0, \nabla^2U) \le AC(y + y_0)^{-2} \le AC(y_0), \qquad \forall y_0 > 0.$$

On the other hand,

$$\int\limits_1^\infty s^{p-1}ds \int\limits_{\mathbf{R}^n} |\nabla^2U(x, s + y_0)|^p dx \le AC \int\limits_1^\infty s^{p-1}(s + y_0)^{-2p}ds \le AC(y_0) < \infty.$$

□

Lemma 4. *Let $0 < p$, and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. If $M_p(y+1, F) \le C$, then V_i (and all its partial derivatives) $\rightrightarrows^{x}_{y\to\infty} 0$, $i = 0, ..., n$, $V_0 = U$.*

Proof. Denote $H = V_i$, and let $B_t(x, t+1)$ be a ball of radius t, centered at $(x, t+1)$. Then Lemma 1 gives

$$|H(x, t + 1)|^p \le A(p)t^{-n-1} \int\limits_{B_t(x-z,s-(t+1))} |H(z, s)|^p dzds \le$$

$$A(p)t^{-n-1}\int\limits_{1}^{2t+1} ds \int\limits_{|z-x|_{\mathbf{R}^n}<t} |H(z,s)|^p dz \le 2A(p)t^{-n}\Big(\sup_{s\ge 1}\int\limits_{\mathbf{R}^n} |H(z,s)|^p dz\Big) \le$$

$$2A(p)t^{-n}\Big(\sup_{t\ge 0}\int\limits_{\mathbf{R}^n} |H(z,t+1)|^p dz\Big) \le ACt^{-n}.$$

Thus, $\sup\limits_{x\in\mathbf{R}^n} |H(x,t+1)| \le ACt^{-n/p}$. □

Lemma 5. *$F=(U,V_1,...,V_n)\in H^p$ if and only if $V_i(x,y)\rightrightarrows^{x}_{y\to\infty} 0$, $i=1,...,n$ and*

$$\int\limits_{\mathbf{R}^n}\Big(\sup_{\eta\ge y}|U(x,\eta)|\Big)^p dx \le C. \tag{11}$$

Proof. The part **only if** is obvious. The part **if** follows by the reasons which are similar to those in Theorem 9 of [4] and the Fatou lemma. □

Lemma 6. *Let $F=(U,V_1,...,V_n)$ be a harmonic vector such that*

$$1)\ M_p(y+1,F)\le C, \qquad 2)\ M_p(y,D_{n+1}U)\le Cy^{-1}. \tag{12}$$

Then

$$\int\limits_{\mathbf{R}^n}\Big(\sup_{y>0}|F(x,y+y_0)|\Big)^p dx \le AC, \qquad \forall y_0>0,\ \ A=A(n,p,y_0). \tag{13}$$

Proof. Let $H=V_i$, $i=0,1,...,n$, $V_0=U$. By the mean-value theorem,

$$\sup_{y>0}|H(x,y+y_0)| \le \sup_{y>0}|H(x,y+y_0+1)| + \sup_{y>0}|D_{n+1}H(x,y+y_0)|.$$

By virtue of Lemma 3 it is enough to show that

$$\int\limits_{\mathbf{R}^n}\Big(\sup_{y>0}|H(x,y+y_0+1)|\Big)^p dx \le AC, \qquad \forall y_0>0,\ \ A=A(n,p,y_0). \tag{14}$$

Let $p>1$. Then (14) follows from 1) and the L^p-boundedness of the maximal function. Let $0<p\le 1$. Then (14) is a consequence of Theorem 5 (take $t=y+y_0+1$ and repeat the arguments similar to those in Lemma 3). □

4. Proof of Theorem 1.

The part **if** is obvious. We show **only if** by proving subsequently that $\nabla^k F\in H^p$, $\nabla^{k-1}F\in H^p$, ..., $F\in H^p$. First of all, Lemmata 4, 5 give $\nabla^k F\in H^p$. Let us show that $\nabla^{k-1}F\in H^p$. It is enough to prove that $D^{k-1}_{n+1}U(x,y)\in H^p$. By the mean-value theorem we have

$$\sup_{y>0}|D^{k-1}_{n+1}U(x,y)| \le \sup_{y>0}|D^{k-1}_{n+1}U(x,y+1)| + \sup_{y>0}|D^{k}_{n+1}U(x,y)|, \tag{15}$$

and it is enough to show that $\sup_{y>0} |D_{n+1}^{k-1} U(x, y+1)| \in L^p$. In fact, we have some more, namely

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}^{k-1} U(x, y+y_0)| \Big)^p dx < C(y_0) \qquad \forall y_0 > 0. \tag{16}$$

To prove (16) we apply the mean-value theorem again, and write

$$\sup_{y>0} |D_{n+1}^{k-1} U(x, y+y_0)| \le \sup_{y>0} |D_{n+1}^{k-1} U(x, y+y_0+1)| + \sup_{y>0} |D_{n+1}^{k} U(x, y+y_0)|.$$

Then $\nabla^k F \in H^p$ implies

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}^{k} U(x, y+y_0)| \Big)^p dx < C(y_0).$$

On the other hand, we have

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}^{k-1} U(x, y+y_0+1)| \Big)^p dx < C(y_0).$$

This follows from Theorem 6, the estimate

$$\int_{\mathbf{R}^n} \Big| \nabla^j F(x, y+y_0+1) \Big|^p dx \le AC(y+y_0)^{-jp}, \qquad 0 \le j \le k, \tag{17}$$

and the reasons which are similar to those in Lemma 3. Thus, we have (16) and $\nabla^{k-1} F \in H^p$.

We repeat the argument and get $\nabla^i F \in H^p$, $0 < i \le k$. Now let $i = 0$. Then the first condition of our theorem, the fact that $\nabla F \in H^p$, and the estimate

$$|U(x, y)| \le |U(x, y+1)| + \sup_{y>0} |D_{n+1} U(x, y)|$$

give $F \in h^p$. Then we may apply Theorem 6 and Lemma 6 to obtain (13). To show that $F \in H^p$ it remains to use

$$\sup_{y>0} |U(x, y)| \le \sup_{y>0} |U(x, y+1)| + \sup_{y>0} |D_{n+1} U(x, y)|.$$

The proof of the theorem is complete.

5. Auxiliary results for the proof of Theorem 2.

Let V_i be components of the harmonic vector $F = (U, V_1, ..., V_n)$. By the mean-value theorem,

$$V_i(x, y) = V_i(x, y+1) - D_{n+1} V_i(x, y+\theta_i), \qquad 0 < \theta_i(x, y) < 1. \tag{18}$$

Lemma 7. *We have*

$$\sup_{y>0} |V_i(x, y)| \le \sup_{y>0} |V_i(x, y+1)| + \sup_{y>0} |D_{n+1} V_i(x, y+\theta_i)|, \tag{19}$$

$$\sup_{y>0} |D_{n+1} V_i(x, y+\theta_i)| \le 2F^+(x), \tag{20}$$

where $i = 0, ..., n$, $V_0 = U$.

Lemma 8. *Let* $F = (U, V_1, ..., V_n) \in H^q$ *and let* θ_i *be as in (18),* $i = 0, ..., n$. *Then*

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y + \theta_i)| \Big)^q dx \leq AC. \tag{21}$$

Proof. By (20) of Lemma 7, we have

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1} V_i(x, y + \theta_i)| \Big)^q dx \leq AC. \tag{22}$$

To get the desired result we apply the Cauchy-Riemann equations $D_i U(x, y) = D_{n+1} V_i(x, y)$. □

The next result uses the observation that $\theta_i > 0$ on a set, controlled by estimate (4).

Lemma 9. *Let* $0 < p < q$ *and let* $F(x, y) = (U(x, y), V_1(x, y), ..., V_n(x, y))$ *be a harmonic vector in* $\mathbf{R}^{n+1}_+$. *Then conditions*

$$1)\ M_p(y + 1, F) \leq C, \qquad 2)\ M_p(y, D_{n+1} U) \leq C y^{-1}, \qquad 3)\ F \in H^q \tag{23}$$

imply $F \in h^p$.

Proof. Let θ_i be as in (18). Due to (8) and condition 1) it is enough to show that $\sup\limits_{y>0} |D_{n+1} V_i(\cdot, y + \theta_i(\cdot, y))| \in L^p(\mathbf{R}^n)$. This will follow from the Cauchy-Riemann equations and

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))| \Big)^p dx < \infty. \tag{24}$$

Thus, we prove (24). Condition 3) and Lemma 1 imply

$$\sup_{x \in \mathbf{R}^n} |D_i U(x, y + \theta_i)| \leq \sup_{x \in \mathbf{R}^n} \frac{AC}{(y + \theta_i)^{1+n/q}} \leq AC(y + \alpha_i(y))^{-1-n/q}, \tag{25}$$

where $\alpha_i(y) = \inf\limits_{x \in \mathbf{R}^n} \theta_i(x, y)$. Define

$$L_i = \{x \in \mathbf{R}^n : \sup_{y>0} |D_i U(x, y + \theta_i)| \leq \sup_{y>0} \frac{AC}{(y + \alpha_i(y))^{1+n/q}} \leq 1\}.$$

Then $\forall x \in CL_i$ (the complement of L_i) we have $\sup\limits_{y>0} |D_i U(x, y + \theta_i)| > 1$. Then

$$\int\limits_{CL_i} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))| \Big)^p dx \leq \int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))| \Big)^q dx \leq AC \tag{26}$$

due to condition 3) and Lemma 8.

We estimate the integral in (24) over L_i. Observe that for fixed $x \in L_i$,

$$\sup_{y>0} |D_i U(x, y + \theta_i(x, y))| \leq \sup_{y>0} |D_i U(x, y + \alpha_i(y))|,$$

and $\alpha_i \equiv \inf_{y>0} \alpha_i(y) > 0$. We put $\gamma = \min_{i=0,\dots,n} \alpha_i > 0$, and take any $0 < y_0 \le \gamma$. Then

$$\int_{L_i} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))|\Big)^p dx \le \int_{\mathbf{R}^n} \Big(\sup_{y>0} |\nabla U(x, y + y_0)|\Big)^p dx \le AC. \tag{27}$$

Indeed, condition 3) and Theorem 6 imply $V_i(x, y) \rightrightarrows^x_{y\to\infty} 0$. The same is true for all partial derivatives of V_i, $i = 0, \dots, n$, $V_0 = U$. Now the second inequality in (27) follows from 2) and Lemma 3. Taking into account (27), (26), we get (24). □

The next result uses the observation that under conditions (23) the supremum $\sup_{y>0} |F(x, y)|$ is reached at the boundary.

Theorem 8. *Let $0 < p < q$ and let $F = (U, V_1, \dots, V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. Then $F \in H^r \; \forall r : p \le r \le q$ if and only if conditions (23) are valid.*

Proof. The part **only if** follows from Theorem 6. We prove **if**. It is enough to show that $F \in H^p$, or $(\sup_{y>0} |F(\cdot, y)|)^p \in L^1(\mathbf{R}^n)$. By the previous lemma $F \in h^p$. Hence, applying the Fatou lemma we have

$$\int_{\mathbf{R}^n} |F(x, 0)|^p dx = \int_{\mathbf{R}^n} \Big(\lim_{y\to 0} |F(x, y)|\Big)^p dx \le \lim_{y\to 0} M_p(y, F) \le C. \tag{28}$$

We claim that $\sup_{y>0} |F(x, y)| = |F(x, 0)|$ and our lemma follows from (28). Since for fixed $x_0 \in \mathbf{R}^n$ the function $WF(x_0, y) \equiv \sup_{\eta \ge y} |F(x_0, \eta)|$ is nonincreasing in y, we have

$$\sup_{y>0} |F(x_0, y)| = \sup_{y>0} \sup_{\eta\ge y} |F(x_0, \eta)| = \lim_{y\to 0} \sup_{\eta \ge y} |F(x_0, \eta)| = |F(x_0, 0)|.$$

Even if $\sup_{y>0} |F(x_0, y)| = |F(x_0, y_0)|$ for some $y_0 > 0$, then $WF(x_0, y) = WF(x_0, y_0)$ for all $0 \le y \le y_0$, and we may put $|F(x_0, y_0)| = |F(x_0, 0)|$. □

Theorem 9. *Let $0 < p < q$, $k \in \mathbf{N}$, and let $F = (U, V_1, \dots, V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. Then $F \in H^r \; \forall r : p \le r \le q$ if and only if*

$$1)\; M_p(y + 1, F) \le C, \qquad 2)\; M_p(y, D^k_{n+1} U) \le C y^{-k}, \qquad 3)\; F \in H^q. \tag{29}$$

Proof. By Corollary 1, and the inverse statement, proved in [9], conditions (23), (29) are equivalent. □

Lemma 10. *Let $p > 0$ and let $F = (U, V_1, \dots, V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$ such that*

$$1)\, M_p(y + 1, F) \le C, \qquad 2)\, M_p(y, U) \le C, \qquad 3)\, |U(x, y)| \le C. \tag{30}$$

Then $F \in H^r$, $r \ge p$.

Proof. By Theorem 7 we have $F \in H^r$, $r > p$. Let $r = p$. By Lemma 4 and Theorem 6 we have $M_p(y, \nabla^k U) \le C y^{-k}$, and we may use Theorem 9. □

Lemma 11. *Let $0 < p < q$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$ such that*

$$(31) \qquad 1)\, M_p(y+1, F) \le C, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \ge y} |D_{n+1}U(x,\eta)| \Big)^q dx \le C.$$

Then $F \in H^q$.

Proof. By the mean-value theorem,

$$\sup_{y>0} |U(x,y)| \le \sup_{y>0} |U(x,y+1)| + \sup_{y>0} |\nabla U(x,y)|,$$

and it is enough to show that $(\sup_{y>0} |U(\cdot, y+1)|)^p \in L^1$. To prove this, we apply Theorem 6, the mean-value theorem again,

$$|U(x,y+1)| \le |U(x,y+2)| + \sup_{y>0} |D_{n+1}U(x,y+1+\theta)|,$$

and observe that conditions of the previous Lemma are satisfied with $y+1$ instead of y. □

6. Proof of Theorem 2.

The proof is given in two lemmata presented below.

Lemma 12. *Let $k \in \mathbf{N}$, $0 < p < q$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. Then F and all its partial derivatives up to order k belong to H^r, $p \le r \le q$, if and only if*

$$(32) \quad 1)\, M_p(y+y_0, F) \le C(y_0) \ \ \forall y_0 > 0, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \ge y} |D^k_{n+1}U(x,\eta)| \Big)^q dx \le C.$$

Proof. The **only if** part is trivial. We prove **if** by induction. Let $k = 1$. We show at first that $F \in H^r$, $p \le r \le q$. By Lemma 11 we have $F \in H^q$, and it is enough to show that $F \in H^p$. To this end, we apply the mean-value theorem and repeate the proof of Lemma 9 begining with (24). As in Lemma 9 we define L_i, and (26) follows from 2). The last estimate in (27) follows from Theorem 6 and Lemma 10. Conditions (30) are satisfied with ∇U instead of F, $D_{n+1}U$ instead of U, and $y + y_0$ instead of y.

To show that all partial derivatives of the first order belong to H^r, $p \le r \le q$ one has to proceed as above by changing ∇U by $\nabla^2 U$, and $D^2_{n+1}U$ by $D_{n+1}U$.

Assume that the statement is true for $k-1$, and we have to prove it for k. By Theorem 6 we have 1) with $D^{k-1}_{n+1}U$ instead of F, and the result follows. □

Lemma 13. *Conditions of the theorem are equivalent to conditions of the previous lemma.*

Proof. It is enough to prove that $M_p(y+1, F) \le C$ and 2) of (32) imply 1) of (32). This will follow from $F \in H^q$. Since $V_i(x,y) \rightrightarrows^x_{y\to\infty} 0$, $i = 1, ..., n$, it is enough to show that $U \in H^q$. We will subsequently show that all $D^{k-1}_{n+1}U, D^{k-2}_{n+1}U, ..., U \in H^q$. In fact, we prove that $D^{k-1}_{n+1}U \in H^q$. The proof of $D^{k-2}_{n+1}U, ..., U \in H^q$ is similar.

Observe that $D_{n+1}^{k-1}U \in H^q$ follows from $D_{n+1}^{k-1}U \in h^q$. Indeed, let $D_{n+1}^{k-1}U \in h^q$. By the mean-value theorem we have

$$\sup_{y>0} |D_{n+1}^{k-1}U(x,y)| \le \sup_{y>0} |D_{n+1}^{k-1}U(x,y+1)| + \sup_{y>0} |D_{n+1}^{k}U(x,y)|,$$

and we may apply Lemma 10 to obtain

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}^{k-1}U(x,y+1)| \Big)^p dx < \infty.$$

Here we use Lemma 10 with $p = q$, $D_{n+1}^{k-1}U(x,y+1)$ instead of $U(x,y)$, $\nabla^{k-1}F$ instead of F. The assumption $D_{n+1}^{k-1}U \in h^q$ and Lemma 1 are used to satisfy the third condition of Lemma 10. The above inequality gives $D_{n+1}^{k-1}U \in H^q$.

Thus, it remains to prove that $D_{n+1}^{k-1}U(x,y) \in h^q$. By the mean-value theorem,

$$|D_{n+1}^{k-1}U(x,y)| \le |D_{n+1}^{k-1}U(x,y+1)| + \sup_{y>0} |D_{n+1}^{k}U(x,y)|,$$

and it is enough to prove that $D_{n+1}^{k-1}U(x,y+1) \in h^q$. Again, by the mean-value theorem,

$$|D_{n+1}^{k-1}U(x,y+1)| \le |D_{n+1}^{k-1}U(x,y+1+1)| + \sup_{y>0} |D_{n+1}^{k}U(x,y)|,$$

but now we may use the assumption $M_p(y+1,F) \le C$ to show that $M_r(y+2, D_{n+1}^{k-1}U) \le C$, $r \ge p$. To this end, we apply Theorem 6, Lemma 1, and take $y+1$ instead of y. □

7. Auxiliary results for the proof of Theorem 3

Lemma 14. *Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}_+^{n+1}$ such that*

$$(33) \qquad 1)\ V_j \rightrightarrows_{y\to\infty}^{x} 0, \qquad 2)\ M_p(y,U) \le C, \qquad 3)\ M_p(y, D_{n+1}U) \le C,$$

$j = 1, ..., n$. Then $F \in H^r$ $\forall r : p < r < np/(n-p)$ provided $0 < p < n$, and $F \in H^r$ $\forall r > p$, provided $p \ge n$.

Proof. Let $0 < p < n$. By 1) it is enough to show that $U \in H^r$ $\forall r : p < r < np/(n-p)$.

Assume at first that $0 < p < n$, $np/(n-p) \le 1$. We have to show (see Theorem 5 and the definition of H^p spaces from [4]) that

$$(34) \qquad \int_0^\infty \int_{\mathbf{R}^n} s^{r-1} |D_{n+1}U(x,s)|^r dx ds = \int_0^1 + \int_1^\infty < C.$$

This can be done by using inequalities proved in [14]:

$$(35) \qquad M_r(y, D_{n+1}U) \le ACy^{-n/p+n/r},$$

$$(36) \qquad M_r(y,F) \le ACy^{-n/p+n/r}, \qquad M_r(y, \nabla U) \le ACy^{-1-n/p+n/r}.$$

The estimate in (35) follows from 3). The estimates in (36) follow from 2).

To estimate the first integral in the right-hand side of (34) we write

$$\int_0^1 s^{r-1}ds\int_{\mathbf{R}^n}|D_{n+1}U(x,s)|^r dx \le AC\int_0^1 s^{r-1}s^{-\frac{nr}{p}+n}ds \le AC,$$

since $r-1-nr/p+n>-1 \iff r<np/(n-p)$. To estimate the second integral in the right-hand side of (34) we use (36),

$$\int_1^\infty s^{r-1}ds\int_{\mathbf{R}^n}|D_{n+1}U(x,s)|^r dx \le AC\int_1^\infty s^{r-1}s^{-r-\frac{nr}{p}+n}ds \le AC,$$

where $r-1-r-nr/p+n<-1$.

Assume now that $1<p<n$, $np/(n-p)>1$. We show that $F\in H^r$, $1<r<np/(n-p)$. Here we use

$$|U(x,y)| \le |U(x,1)| + \int_0^1 |D_{n+1}U(x,\eta)|d\eta. \tag{37}$$

The first inequality in (36) implies

$$M_r(1,F)\le AC. \tag{38}$$

Since $F(x,1)$ is bounded, (38) is true $\forall r>p$, and it is enough to show that the second term in the right-hand side of (37) belongs to H^r, $1<r<np/(n-p)$. By Minkowsi's inequality we have

$$\Big(\int_{\mathbf{R}^n}\int_0^1 |D_{n+1}U(x,y)|^r dxdy\Big)^{1/r} \le \int_0^1\Big(\int_{\mathbf{R}^n}|D_{n+1}U(x,y)|^r dx\Big)^{1/r}dy \le$$

$$\int_0^1 y^{-n/p+n/r}dy \le AC, \qquad -n/p+n/r>-1.$$

If $p<1$, $np/(n-p)>1$, then one has to split $(p,np/(n-p))$ into two intervals $(p,1]$, $(1,np/(n-p))$, and to repeat the previous parts of the proof.

Finally, let $p\ge n$. We take $r>p\ge n\ge 1$, i.e. $r>1$ and we may use Minkowski's inequality. □

Remark. *The previous lemma is sharp in the case* $0<p\le 1$. *There exists a harmonic vector* F *satisfying (33) such that* $F\notin H^p$, $F\notin H^{np/(n-p)}$.

Proof. We take for simplicity $n=1$, $p=1/2$, $q=np/(n-p)=1$. The multidimensional example can be constructed by using the Poisson kernel. Consider a harmonic vector

$$U(x,y)=\frac{x}{x^2+y^2}-\frac{x}{x^2+(y+1)^2}, \qquad V(x,y)=-\Big(\frac{y}{x^2+y^2}-\frac{y+1}{x^2+(y+1)^2}\Big).$$

Then

$$|F(x,y)| = \frac{1}{\sqrt{x^2+y^2}\sqrt{x^2+(y+1)^2}}, \qquad \sup_{y>0}|F(x,y)| = \frac{1}{|x|\sqrt{x^2+1}} \le \frac{\sqrt{2}}{|x|(|x|+1)}.$$

First of all, $F \in H^r \; \forall r : \; 1/2 < r < 1$,

$$\int_{-\infty}^{\infty} (\sup_{y>0}|F(x,y)|)^r dx \le A(r) \int_{-\infty}^{\infty} \frac{dx}{(|x|(|x|+1))^r} < C.$$

On the other hand, it is obvious that $F \notin H^1$,

$$\int_{-\infty}^{\infty} \sup_{y>0}|F(x,y)|dx = \int_{-\infty}^{\infty} \frac{dx}{|x|\sqrt{x^2+1}} \ge \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{x} = +\infty,$$

and $F \notin H^{1/2}$,

$$\int_{-\infty}^{\infty} \sqrt{\sup_{y>0}|F(x,y)|}dx = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{|x|\sqrt{x^2+1}}} \ge 2^{-1/4} \int_1^{\infty} \frac{dx}{x} = +\infty.$$

It remains to show that $M_{1/2}(y,U) \le C$, $M_{1/2}(y, D_{n+1}U) \le C$. We have

$$M_{1/2}(y,U) = \int_{-\infty}^{\infty} \sqrt{|U(x,y)|}dx = 2\int_0^{\infty} \sqrt{\frac{x(2y+1)}{(x^2+y^2)(x^2+(y+1)^2)}}dx \le$$

$$2\sqrt{2}\int_0^{\infty} \sqrt{\frac{xy}{(x^2+y^2)^2}}dx + 2\int_0^{\infty} \frac{dx}{\sqrt{x}\sqrt{x^2+1}} \le C,$$

where in the first integral of the above inequality one has to make a substitution $x/y = t$. Now,

$$D_{n+1}U(x,y) = \frac{2xy}{(x^2+y^2)^2} - \frac{2x(y+1)}{(x^2+(y+1)^2)^2}$$

and the estimate $M_{1/2}(y, D_{n+1}U) \le C$ is similar. □

Lemma 14 can be generalized to the case of partial derivatives of any order.

Lemma 15. *Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$ such that*

(39) $\quad 1)\, V_j \rightrightarrows^{x}_{y\to\infty} 0,\; j = 1, ..., n, \qquad 2)\, M_p(y,U) \le C, \qquad 3)\, M_p(y, D^k_{n+1}U) \le C$

for some $k \in \mathbf{N}$.

a) If $0 < kp < n$, then $F \in H^r \; \forall r : \; p < r < np/(n-kp)$. If $kp \ge n$, then $F \in H^r$ $\forall r \ge p$.

b) Let $k > 1$, $1 \le m \le k-1$, $m \in \mathbf{N}$. If $0 < (k-m)p < n$, then all partial derivatives of F of the order m belong to $H^r \; \forall r : \; p < r < np/(n-(k-m)p)$. If $(k-m)p \ge n$, then all partial derivatives of F of the order m belong to $H^r \; \forall r \ge p$.

Proof. The proof is a consequence of Lemma 14 and Theorem 2. For convenience of the reader we give it here.

We begin with b), $0 < (k-m)p < n$. Let $m = k-1$. Then 2) and Lemma 2 imply $M_p(y, \nabla^{k-1}U) \leq Cy^{-(k-1)}$, and we may follow the proof of Lemma 14 with $D^{k-1}_{n+1}U$ instead of U. To estimate the integral, similar to (34), we use $M_r(y, \nabla^{k-1}U) \leq Cy^{-(k-1)-n/p+n/r}$ instead of the second inequality in (36). We conclude that all partial derivatives of the order $k-1$ belong to H^r, $p < r < np/(n-p)$.

Now let $m = k-2$. By the previous step, $D^{k-1}_{n+1} \in H^{\frac{np}{n-p}-\epsilon}$, where $\epsilon > 0$ is small enough. We repeat the proof of Lemma 14 with $\frac{np}{n-p} - \epsilon$ instead of p. We have all partial derivatives of the order $k-2$ in H^r, $\frac{np}{n-p} - \epsilon < r < \frac{n(np/(n-p)-\epsilon)}{n-np/(n-p)+\epsilon}$. Letting $\epsilon \to 0$, we have $\frac{np}{n-p} \leq r < \frac{np}{n-2p}$. It remains to show that all partial derivatives of the order $k-2$ belong to H^r, $p < r < \frac{np}{n-p}$. Since all partial derivatives of the order $k-1$ belong to H^r, $p < r < np/(n-p)$, we may use Theorem 2 with $q = \frac{np}{n-p} - \epsilon$, and $k-1$ instead of k. Thus, we get b) for $m = k-2$, provided $0 < (k-m)p < n$. The proof of the cases $m = k-3, k-4, ..., 1$, $0 < (k-m)p < n$, is similar.

We prove b), the case $(k-m)p \geq n$. If $m = k-1$, then the result follows from Lemma 14 (repeat the proof with $D^{k-1}_{n+1}U$ instead of U).

Let $m = k-2$. We write (37) with $D^{k-2}_{n+1}U$ instead of U, and observe that all partial derivatives of the order $k-2$ belong to H^r $\forall r \geq p$, provided $p \geq n$. To get the case $2p \geq n$ we apply Theorem 2. Thus, we get b) for $m = k-2$, provided $(k-m)p \geq n$. The proof of the cases $m = k-3, k-4, ..., 1$, $(k-m)p \geq n$, is similar. The proof of b) is complete.

We show a) by induction on k. For $k = 1$, a) is just Lemma 14. Assume that a) is true for $k-1$, $0 < (k-1)p < n$. To prove the statement for k, $0 < kp < n$, it is enough to show that $F \in H^r$, $\frac{np}{n-(k-1)p} < r < \frac{np}{n-kp}$. To this end, we observe that b) and the induction hypothesis force the first derivative of F to be in H^r, $p < r < \frac{np}{n-(k-1)p}$. Hence, we may apply Lemma 14 (with $\frac{np}{n-(k-1)p} - \epsilon$ instead of p) to get $F \in H^r$, $\frac{np}{n-(k-1)p} < r < \frac{np}{n-kp}$.

Now, let a) be true for $k-1$, $(k-1)p \geq n$. To prove the statement for k, $kp \geq n$, we use Theorem 2. □

8. Proof of Theorem 3

Now we prove Theorem 3. We prove a). By Theorem 1 and Lemma 15, we have $F \in H^r, \forall r : p \leq r < np/(n-kp)$. We have to prove that $F \in H^{np/(n-kp)}$. We prove this in two steps. At first we assume that

$$(40)\quad I(a, \frac{np}{n-kp}) \equiv \int\limits_{\mathbf{R}^{n+1}_+} y^a |F(x,y)|^{np/(n-kp)} dxdy < \infty, \qquad \text{for some } -1 < a < 0,$$

and prove that (40) implies $F^+(x) \in L^{np/(n-kp)}$. Then we prove (40).

Let (40) be true. Define $E = \{x \in \mathbf{R}^n : F^+(x) \leq 1\}$, and let CE denote the complement of E. Since $F \in H^r$, $p \leq r < np/(n-kp)$, we have $m(CE) < \infty$. If

$F \notin L^{np/(n-kp)}(CE)$, then

$$I(a, \frac{np}{n-kp}) \geq \int_0^1 y^a dy \int_{\mathbf{R}^n} |F(x,y)|^{np/(n-kp)} dx \geq \int_0^1 dy \int_{\mathbf{R}^n} |F(x,y)|^{np/(n-kp)} dx.$$

But the last integral may be arbitrary large, a contradiction.

It remains to prove (40). We split $I(a, \frac{np}{n-kp})$ into two parts, $\left(\int_0^1 + \int_1^\infty\right) dy$. Since $M_r(y,F) \leq C$, $p \leq r < np/(n-kp)$, we have $M_r(y,F) \leq ACy^{-n/p+n/r}$ for $r > p$. Let $r = np/(n-kp)$, then $-nr/p + n = -nkp/(n-kp) < np/(n-kp)$. Chosing $a \in (-1,0)$ such that $a - np/(n-kp) < -1$, we obtain

$$\int_1^\infty y^a dy \int_{\mathbf{R}^n} |F(x,y)|^{np/(n-kp)} dx \leq AC \int_1^\infty y^a y^{-\frac{n^2 p}{p(n-kp)}+n} dy < \infty.$$

To estimate $\int_0^1 y^a dy \int_{\mathbf{R}^n} |F(x,y)|^{np/(n-kp)} dx$, we write

$$\int_0^1 y^a dy \int_E |F(x,y)|^{np/(n-kp)} dx + \int_0^1 y^a dy \int_{CE} |F(x,y)|^{np/(n-kp)} dx.$$

The first integral is obviously finite. To get a bound for the second one we fix $a : a - \frac{np}{n-kp} < -1$ as above, and choose $r < \frac{np}{n-kp}$ such that $-(\frac{np}{n-kp} - r)\frac{n}{r} + a > -1$. Then we use $\sup_{x \in \mathbf{R}^n} |F(x,y)| \leq ACy^{-n/r}$ to obtain

$$\int_0^1 y^a dy \int_{CE} |F(x,y)|^{np/(n-kp)} dx \leq AC \int_0^1 y^{a-(\frac{np}{n-kp}-r)\frac{n}{r}} dy \int_{\mathbf{R}^n} (F^+(x))^r dx \leq$$

$$AC \int_0^1 y^{a-(\frac{np}{n-kp}-r)\frac{n}{r}} dy \leq AC.$$

Thus, we have a).

We prove b). By Theorem 1 and Lemma 15, all partial derivatives of F of the order m belong to H^r, $\forall r : p \leq r < np/(n-(k-m)p)$. We have to prove that all partial derivatives of F of the order m belong to $H^{np/(n-(k-m)p)}$. The proof of this is similar to the proof of a) with obvious changes, one has only take $\nabla^m F$ instead of F, and $np/(n-(k-m)p)$ instead of $np/(n-kp)$.

The proof of Theorem 3 is complete.

9. Proof of Theorem 4.

Lemma 16. *Let $\alpha > 0$, $\beta > -1$, $p > 0$ and let $W(x, y) \geq 0$ in $\mathbf{R}_+^{n+1}$ be such that W^p be subharmonic and*

$$I(\alpha, p, \beta) = \int\limits_{\mathbf{R}_+^{n+1}} s^{\alpha p+\beta} W^p(x, s) ds < \infty.$$

Then the function $W_\alpha(x, y) \equiv \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty s^{\alpha-1} W(x, s + y) ds$ is subharmonic, and $\forall y \geq y_0 > 0$ we have

$$\int\limits_{\mathbf{R}^n} (W_\alpha(x, y))^p dx \leq A(\alpha, n, p, \beta, y_0) I(\alpha, p, \beta).$$

In particular $W_\alpha \in S^p(\mathbf{R}_+^{n+1})$.

Proof. The proof is based on the arguments, which are similar to those in [5], Lemma 16 and Theorem 3.

We take $U = W^p(x, t)$, $q = \alpha p + \beta + 1$ in Lemma 16, [5]. Since W^p is subharmonic, and $I(\alpha, p, \beta)$ is convergent, we may use the proof of Theorem 3, [5], to conclude that the integral defining $W_\alpha(x, y)$ is convergent and $W_\alpha(x, y)$ is subharmonic. There exist $f \geq 0$ and a constant $A(n, p, \beta, \alpha)$ such that

$$\|f\|_1 \leq A(n, p, \beta, \alpha) I(\alpha, p, \beta), \qquad W(x, s+y) \leq A(n, p, \beta, \alpha)(s+y)^{-\alpha-(1+\beta)/p} f^{1/p}(x),$$

and we have

$$W_\alpha(x, y) \leq A(n, p, \beta, \alpha) y_0^{-\alpha-(1+\beta)/p} f^{1/p}(x), \qquad \forall y \geq y.$$

□

The proof of the next lemma can be obtained by the reasons which are similar to those in [10].

Lemma 17. *Let $p > 0$, $\alpha > -1$, $m \in \mathbf{N}$, and F be a harmonic vector such that $F \rightrightarrows_{y\to\infty}^{x} 0$, $I(\alpha, m, p) \equiv \int\limits_{\mathbf{R}_+^{n+1}} t^{\alpha+mp} |D_{n+1}^m U(x, t)|^p dx dt < \infty$. Then*

$$\int\limits_{\mathbf{R}_+^{n+1}} t^\alpha |F(x, t)|^p dx dt \leq A(n, p, \alpha) C I(\alpha, m, p).$$

As a consequence of two previous lemmata we have

Lemma 18. *Let $p > 0$, $\alpha > -1$, and let F be a harmonic vector such that*

$$\int\limits_{\mathbf{R}_+^{n+1}} t^\alpha |F(x, t)|^p dx dt < \infty.$$

Then all tensor coordinates of the rank $k \geq 1$ of V_i, $i = 0, ..., n$, $V_0 = U$, belong to S^p. In particular, $\left(\sup\limits_{y>0} |F(\cdot, y + y_0)|\right)^p \in L^1$ $\forall y_0 > 0$.

Now we prove Theorem 4. We start with a). Let $np/(n-p) < r < 1$. We claim that

$$(41)\qquad \int\limits_{\mathbf{R}^{n+1}_+} y^{r-1}|D_{n+1}U(x,y)|^r dxdy = \Big(\int\limits_0^1 + \int\limits_1^\infty\Big) y^{r-1}dy \int\limits_{\mathbf{R}^n} |D_{n+1}U(x,y)|^r dx < \infty.$$

The first integral in the right-hand side of the above equality is obviously finite for all $r > 0$. Since $M_r(y, D_{n+1}U) \le ACy^{-n/p+n/r}$, $r > p$, the second integral is also finite. By Theorem 5 we have $M_r(y,U) \le C$, $r > np/(n-p)$. The mean-value theorem, and condition 3) imply the boundedness of U. By Theorem 7, $U \in H^r$, $1 > r > np/(n-p)$.

Now, let $r > 1$, and let $\gamma > 0$ be such that $r - 1 + \gamma - nr/p + n < -1$, together with $r - 1 - nr/p + n < -1$. Then the integral

$$\int\limits_{\mathbf{R}^{n+1}_+} y^{r+\gamma-1}|D_{n+1}U(x,y)|^r dxdy < \infty,$$

and Lemma 17 (with $m = 1$, $\alpha = \gamma - 1$) gives

$$\int\limits_{\mathbf{R}^{n+1}_+} y^{\gamma-1}|D_{n+1}U(x,y)|^r dxdy < \infty.$$

Since $\gamma - 1 > -1$, we apply Lemma 18 to obtain $\Big(\sup\limits_{y>0}|F(\cdot, y+1)|\Big)^r \in L^1$, $r > np/(n-p)$. By Theorem 7, $D_{n+1}U \in H^r$, and the mean-value theorem implies $U \in H^r$.

We prove b). Let $n = 1$ and consider

$$U(x,y) = \frac{1/2\log(x^2+(y+2)^2)}{1/4\log^2(x^2+(y+2)^2) + \arctan^2(x/(y+2))},$$

$$V(x,y) = \frac{\arctan(x/(y+2))}{1/4\log^2(x^2+(y+2)^2) + \arctan^2(x/(y+2))}.$$

It is clear that U, V satisfy all conditions of the Theorem, nevertheless $F \notin H^r$, provided $r > 0$.

The proof of Theorem 4 is complete.

Observe that all conditions of the Theorem are essential and inependent one on another. Indeed, let $n = 1$, $p \ge 1$,

$$U(x,y) = \log(x^2 + (y+1)^2), \qquad V(x,y) = -\arctan\frac{x}{y+1}.$$

This functions satisfy all conditions of the theorem, but the first one, and $F \notin H^r$, $r > 0$.

Now, let $n = 1$, $p = 1/2$,

$$U(x,y) = \frac{x}{x^2+y^2}, \qquad V(x,y) = -\frac{y}{x^2+y^2}.$$

We have 1) and 2), but not 3), and $F \notin H^r$, $r > 0$.

Let $n = 2$, $r = \sqrt{x_1^2 + x_2^2 + (y+1)^2}$,

$$F = (U, V_1, V_2), \qquad V_1 = \frac{1}{r}, \qquad V_2 = \frac{x_2}{r(r+x_1)}, \qquad U = \frac{y+1}{r(r+x_1)}.$$

We have 1) and 3), but not 2), and $F \notin H^r$, $r > 0$.

Finally, we note that one can not conclude that $F \in H^r$, $r = np/(n-p)$. Indeed, let $n = 1$, $p = 1/2$, $np/(n-p) = 1$. Then

$$U = \frac{x}{x^2 + (y+1)^2}, \qquad V = \frac{y+1}{x^2 + (y+1)^2}$$

satisfy all conditions of the Theorem , but $F \in H^r$, $r > 1$, $r > np/(n-p)$ and $F \notin H^1$.

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