ON POLYTOPES WITH CONGRUENT PROJECTIONS OR SECTIONS

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Abstract. Let $2 \le k \le d-1$ and let P and Q be two convex polytopes in $\mathbb{E}^d$. Assume that their projections, $P|H$, $Q|H$, onto every k-dimensional subspace H, are congruent. In this paper we show that P and Q or P and $-Q$ are translates of each other. We also prove an analogous result for sections by showing that $P = Q$ or $P = -Q$, provided the polytopes contain the origin in their interior and their sections, $P \cap H$, $Q \cap H$, by every k-dimensional subspace H, are congruent.

1. Introduction

In this paper we address the following problems (cf., for example, [Ga, Problem 3.2, p. 125 and Problem 7.3, p. 289]).

Problem 1. *Let $2 \le k \le d-1$. Assume that K and L are convex bodies in $\mathbb{E}^d$ such that the projections $K|H$ and $L|H$ are congruent for all $H \in \mathcal{G}(d,k)$. Is K a translate of $\pm L$?*

Problem 2. *Let $2 \le k \le d-1$. Assume K and L are star bodies in $\mathbb{E}^d$ such that the sections $K \cap H$ and $L \cap H$ are congruent for all $H \in \mathcal{G}(d,k)$. Is $K = \pm L$?*

Here we say that $K|H$, the projection of K onto H, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k,H)$ in H such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(d,k)$ stands for the Grassmann manifold of all k-dimensional subspaces in $\mathbb{E}^d$.

Golubyatnikov has obtained some partial answers to Problem 1 in the case of *direct* rigid motions, i.e., when the orthogonal group $O(k)$ is replaced by the special orthogonal group $SO(k)$. In particular, he proved that the answer is affirmative, provided $k = 2$ and none of the projections of the bodies have symmetries with respect to rotations (in other words, the only direct rigid motion taking $K|H$, $L|H$ onto themselves is the identity). We refer the reader to [Go], [Ga] (pp. $100-110$), [Ha] (pp. $126-127$), [R1], [R2], [R3] and [ACR], for history and partial results related to the above problems.

In this paper we give an affirmative answer to Problems 1 and 2 in the class of convex polytopes. Our first result is

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Theorem 1. *Let $2 \le k \le d-1$ and let P and Q be two convex polytopes in $\mathbb{E}^d$ such that their projections $P|H$, $Q|H$, onto every k-dimensional subspace H, are congruent. Then there exists $b \in \mathbb{E}^d$ such that $P = Q + b$ or $P = -Q + b$.*

If projections of the bodies are *directly* congruent, Golubyatnikov proved Theorem 1 in the case $k = 2$ (the result is contained in the proof of Theorem 2.1.2, p. 19; cf. [Go], p. 17, Lemma 2.1.4), and in the case $k = 3$ under the additional assumption that none of the 3-dimensional projections have rigid motion symmetries (cf. [Go], p. 48, Theorem 3.2.1); see also [ACR], where the assumption on symmetries was relaxed.

Our second result is

Theorem 2. *Let $2 \le k \le d-1$ and let P and Q be two convex polytopes in $\mathbb{E}^d$ containing the origin in their interior. Assume that their sections, $P \cap H$, $Q \cap H$, by every k-dimensional subspace H, are congruent. Then $P = Q$ or $P = -Q$.*

If sections of the bodies are *directly* congruent Theorem 2 is known in the case $k = 2$ (cf. [AC]), and in the case $k = 3$ (cf. [ACR]) under the additional assumptions that diameters of one of the polytopes contain the origin and certain sections related to the diameters have no rigid motion symmetries.

We remark that in both Theorems it is enough to assume that only one of the bodies is a convex polytope. This follows from a result of Klee [Kl], who showed that a set in $\mathbb{E}^d$ must be a convex polytope, provided all of its projections on k-dimensional subspaces are convex polytopes. It follows by duality (see [Ga], formula (0.38), p. 22) that if all sections of a convex body containing the origin in its interior are polytopes, then the body is a polytope.

The paper is organized as follows. In the next section we recall some definitions for convenience of the reader. We also prove some auxiliary Lemmata that will be used in Sections 3 and 4, where we prove Theorems 1 and 2.

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1.1. **Notation.** We denote by $S^{d-1} = \{x \in \mathbb{E}^d : |x| = 1\}$ the set of all unit vectors in the Euclidean space $\mathbb{E}^d$, and by $S^{d-1}_{\zeta} = \{x \in S^{d-1} \colon x \cdot \zeta < 0\}$ the open hemisphere in the direction $\zeta \in S^{d-1}$; O stands for the origin of $\mathbb{E}^d$. We write that line l_1 is parallel to line l_2 as $l_1 \parallel l_2$. For any unit vector $\xi \in S^{d-1}$ we let $\xi^{\perp}$ to be the orthogonal complement of ξ in $\mathbb{E}^d$, i.e., the set of all $x \in \mathbb{E}^d$ such that $x \cdot \xi = 0$; here $x \cdot \xi$ stands for a usual scalar product of x and $\xi \in \mathbb{E}^d$. The notation for the orthogonal group $O(k)$ and the special orthogonal group $SO(k)$ is standard. The notation $\varphi_{\xi} \in O(d-1, \xi^{\perp})$ will be used for an orthogonal transformation acting in $\xi^{\perp}$. We agree to denote by xy the closed interval connecting x and y, i.e., all points of the form $ty + (1-t)x, t \in [0,1]$; the shortest arc of the unit circle joining the points x and y on S^{d-1} will be denoted by $[xy]$; $span(a_1, a_2, \dots, a_m)$ stands for the m-dimensional subspace that is a linear span of the linearly independent vectors $a_1, \dots, a_m$; $span(A)$ is the linear subspace of smallest dimension containing a set $A \subset \mathbb{E}^d$. Also, for any set $A \subset \mathbb{E}^d$ the notation $A_\xi = A|\xi^\perp$ is used for the projection of A onto $\xi^\perp$. The *shadow boundary* of a convex polytope P in direction ξ will be denoted by $\partial_\xi P$, i.e., $\partial_\xi P = \{x \in P : x_\xi \in \partial P_\xi\}$; here ∂P_ξ stands for the boundary of the projection P_ξ. Given a set B, $\text{conv}(B)$ denotes the smallest convex set containing B.

2. Definitions and Auxiliary Results

A set $A \subset \mathbb{E}^d$ is convex if for any two points x and y in A the closed line segment xy joining them is in A. A convex *body* $K \subset \mathbb{E}^d$ is a compact convex set with a non-empty interior (with respect to $\mathbb{E}^d$). A *convex polytope* $P \subset \mathbb{E}^d$ is a convex body that is the convex hull of finitely many points (called the vertices of P).

Fix $1 \le k \le d$. We say that a convex set B is a *k-dimensional convex polytope* if there exists an affine k-dimensional subspace M of $\mathbb{E}^d$ such that $B \subset M$ and B is a convex polytope relative to M (i.e., the interior of B is taken with respect to M).

We will say that a subset D of an open hemisphere S^{d-1}_ζ, $z\eta \in S^{n-1}$, is *geodesically convex*, if for any two points x and y in D the arc of the unit circle $[xy]$ joining them is in D.

We define a *rigid motion* T_ξ acting in $\xi^\perp$, $T_\xi : \xi^\perp \to \xi^\perp$, as a composition of an orthogonal transformation φ_ξ and a translation by $a_\xi \in \xi^\perp$, $T_\xi(x) = \varphi_\xi(x) + a_\xi$ for all $x \in \xi^\perp$.

The supporting hyperplane G to a convex body K is defined as the hyperplane having common points with K and such that K lies in one of the two closed half-spaces with the boundary G. The *support function* of a convex body is defined as $h_K(\xi) = \max\limits_{x \in K} x \cdot \xi$ for all $\xi \in \mathbb{E}^d$.

If L is a convex body containing the origin in the interior, its *radial function* is defined as $\rho_L(\xi) = \max\{\lambda > 0 : \lambda\xi \in L\}$ for all $\xi \in \mathbb{E}^d$. Since h_K and ρ_K are homogeneous functions of degrees 1 and -1 correspondingly, it is enough to consider their values for $\xi \in S^{d-1}$, where both functions are continuous.

Our first auxiliary result will be used in the proof of Theorem 1.

Lemma 2.1. *Consider four points $A, B, C, D \in \mathbb{E}^d$, such that for an open set $U \subset S^{d-1}$ of directions ξ, $|A_\xi B_\xi| = |C_\xi D_\xi| \neq 0$. Then the intervals AB and CD are parallel and have equal length.*

Proof. Since any parallel translation preserves the length of the projections of the intervals, we may assume that $A = C = O$. Let a and b be the unit vectors of directions of AB and CD, and t and s be their lengths. In this case, the projections of B and D onto $\xi^\perp$ can be found as

$$B_\xi = ta - (\xi \cdot ta)\xi \quad \text{and} \quad D_\xi = sb - (\xi \cdot sb)\xi, \qquad \forall \xi \in U.$$

Our goal is to prove that $t = s$ and $a = b$ or $a = -b$.

The condition on the equality of the lengths of projections can be written as

$$t|a - (\xi \cdot a)\xi| = s|b - (\xi \cdot b)\xi| \qquad \forall \xi \in U,$$

or

$$t^2 - s^2 = t^2(\xi \cdot a)^2 - s^2(\xi \cdot b)^2 = (\xi \cdot (ta - sb))(\xi \cdot (ta + sb)) \qquad \forall \xi \in U.$$

We claim that the right-hand side of the above identity is not an identically constant function of ξ on the spherical cap U, unless $ta - sb = 0$ or $ta + sb = 0$. Indeed, if the claim is false, then dividing the above identity by the lengths of vectors $ta - sb$, $ta + sb$, we see that the function

$$f(\xi) = (\xi \cdot v)(\xi \cdot w), \quad v = \frac{ta - sb}{|ta - sb|}, \quad w = \frac{ta + sb}{|ta + sb|},$$

is identically constant (equal to $\frac{t^2 - s^2}{|ta-sb||ta+sb|}$) on U. For $c_1, c_2 \in (-1, 1)$ consider the level sets

$$L_v(c_1) = \{\xi \in S^{d-1} : \xi \cdot v = c_1\}, \quad L_w(c_2) = \{\xi \in S^{d-1} : \xi \cdot w = c_2\},$$

of functions $\xi \to \xi \cdot v$ and $\xi \to \xi \cdot w$. It is clear that $L_v(c_1)$, $L_w(c_2)$ are the corresponding $(d-2)$-dimensional sub-spheres of S^{d-1}. Since $v \neq \pm w$, $L_v(c_1) \neq L_w(c_2)$ $\forall c_1,\ c_2 \in (-1, 1)$, and $L_v(c_1) \cap L_w(c_2)$ is (at most) an $(d-3)$-dimensional subsphere of S^{d-1}. Taking $c_1, c_2 \in (-1, 1)$ such that $L_v(c_1) \cap L_w(c_2) \cap U \neq \emptyset$, and changing ξ in $U \cap (L_v(c_1) \setminus L_w(c_2))$, we see that $\xi \cdot v$ remains constant while $\xi \cdot w$ does not. Thus, $f(\xi)$ is not constant on U, and we obtain a contradiction. The claim is proven, and the proof of the Lemma is finished. □

The next statement shows that one may disregard the set of all directions for which at least two facets of the projections are orthogonal. Here we say that two affine subspaces of co-dimension one are orthogonal if their normal vectors are orthogonal; two facets of a polytope are orthogonal if the affine subspaces containing them are.

Lemma 2.2. *Let α and β be two non-parallel $(d-2)$-dimensional subspaces in $\mathbb{E}^d$, such that $\dim(\alpha \cap \beta) = d - 3$. Then the set of directions $\xi \in S^{d-1} \setminus (\alpha \cup \beta)$, for which α_ξ is orthogonal to β_ξ, is a nowhere dense subset of S^{d-1}.*

Proof. Assume that $\alpha \cap \beta = span\{c_1, c_2, \dots, c_{d-3}\}$ and $\alpha = span\{a, c_1, \dots, c_{d-3}\}$, $\beta = span\{b, c_1, \dots, c_{d-3}\}$, where a, b, and c_j are linearly independent vectors in $\mathbb{E}^d$, $j = 1, \dots, d-3$, $a \neq b$. By condition of the lemma, we have $\alpha_\xi = span\{a_\xi, (c_1)_\xi, \dots, (c_{d-3})_\xi\}$, $\beta_\xi = span\{b_\xi, (c_1)_\xi, \dots, (c_{d-3})_\xi\}$, $\alpha_\xi \cap \beta_\xi = span\{(c_1)_\xi, \dots, (c_{d-3})_\xi\}$. Here $(c_i)_\xi = c_i - (c_i \cdot \xi)\xi$, $i = 1, \dots, d-3$, and $a_\xi = a - (a \cdot \xi)\xi$, $b_\xi = b - (b \cdot \xi)\xi$.

Let n_{α_ξ} and n_{β_ξ} be the normal vectors to α_ξ and β_ξ in $\xi^\perp$. If $\alpha_\xi \perp \beta_\xi$, then $n_{\alpha_\xi} \perp n_{\beta_\xi}$, and, in the three-dimensional case, the condition of orthogonality $n_{\alpha_\xi} \cdot n_{\beta_\xi} = 0$ can be written as

$$a_\xi \cdot b_\xi = (a - \xi(a \cdot \xi)) \cdot (b - \xi(b \cdot \xi)) = a \cdot b - (a \cdot \xi)(b \cdot \xi) = 0.$$

To write out the condition in the case $d \geq 4$, we consider a linear transformation $A_\xi \in O(d)$, such that $A_\xi \xi = e_d = (0, \dots, 0, 1)$. For all $\xi \in S^{d-1}$, $\xi \neq e_d$, we identify A_ξ with the corresponding matrix

$$A_\xi = \begin{pmatrix} -1 + \frac{\xi_1^2}{1-\xi_d} & \frac{\xi_1 \xi_2}{1-\xi_d} & \dots & \frac{\xi_1 \xi_{d-1}}{1-\xi_d} & -\xi_1 \\ -\frac{\xi_1 \xi_2}{1-\xi_d} & 1 - \frac{\xi_2^2}{1-\xi_d} & \dots & -\frac{\xi_2 \xi_{d-1}}{1-\xi_d} & \xi_2 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{\xi_1 \xi_{d-1}}{1-\xi_d} & -\frac{\xi_{d-1} \xi_2}{1-\xi_d} & \dots & 1 - \frac{\xi_{d-1}^2}{1-\xi_d} & \xi_{d-1} \\ \xi_1 & \xi_2 & \dots & \xi_{d-1} & \xi_d \end{pmatrix},$$

(if we treat the i-th row r_i of A_ξ as a d-dimensional vector, then $r_i \cdot r_j = \delta_{ij}$, i, j=1, $\dots$, d). If $\xi = e_d$ we put $A_\xi = I$.

Consider now the vectors $\tilde{a} = \tilde{a}(\xi) = A_\xi a_\xi, \tilde{b} = \tilde{b}(\xi) = A_\xi b_\xi, \tilde{c}_i = \tilde{c}_i(\xi) = A_\xi (c_i)_\xi, \tilde{n}_{\alpha_\xi} = A_\xi n_{\alpha_\xi}, \tilde{n}_{\beta_\xi} = A_\xi n_{\beta_\xi}$. The condition of the normals being orthogonal is

$$\tilde{n}_{\alpha_\xi} \cdot \tilde{n}_{\beta_\xi} = n_{\alpha_\xi} \cdot n_{\beta_\xi} = 0 \qquad \xi \in S^{d-1} \setminus (\alpha \cup \beta), \tag{1}$$

where the normal vectors $\tilde{n}_{\alpha_\xi}, \tilde{n}_{\beta_\xi} \in e_d^\perp$ can be found as the generalized vector products,

$$\tilde{n}_{\alpha_\xi} = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{d-1} \\ \tilde{a}^1 & \tilde{a}^2 & \tilde{a}^3 & \dots & \tilde{a}^{d-1} \\ \tilde{c}_1^1 & \tilde{c}_1^2 & \tilde{c}_1^3 & \dots & \tilde{c}_1^{d-1} \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{c}_{d-3}^1 & \tilde{c}_{d-3}^2 & \tilde{c}_{d-3}^3 & \dots & \tilde{c}_{d-3}^{d-1} \end{vmatrix},$$

$$\tilde{n}_{\beta_\xi} = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{d-1} \\ \tilde{b}^1 & \tilde{b}^2 & \tilde{b}^3 & \dots & \tilde{b}^{d-1} \\ \tilde{c}_1^1 & \tilde{c}_1^2 & \tilde{c}_1^3 & \dots & \tilde{c}_1^{d-1} \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{c}_{d-3}^1 & \tilde{c}_{d-3}^2 & \tilde{c}_{d-3}^3 & \dots & \tilde{c}_{d-3}^{d-1} \end{vmatrix}.$$

Here $\tilde{a}^j, \tilde{b}^j$ stand for the j-th coordinate of the vectors $\tilde{a}$ and $\tilde{b}$, $\tilde{c}_i^j$ is the j-th coordinate of the vector $\tilde{c}_i$, and e_j, $j = 1, \dots, d-1$, are the vectors of the standard basis in $e_d^\perp$.

Notice that the j-th coordinate of the normal $\tilde{n}_{\alpha_\xi}$ is a rational function $R_j^\alpha(\xi) = \frac{P_j^\alpha(\xi)}{(1-\xi_d)^{d-2}}$ of the coordinates of ξ, where $P_j^\alpha(\xi)$ is a polynomial in coordinates of ξ. Similarly for $\tilde{n}_{\beta_\xi}$, i.e.,

$$\tilde{n}_{\alpha_\xi} = (R_1^\alpha(\xi), \dots, R_{d-1}^\alpha(\xi)), \quad \tilde{n}_{\beta_\xi} = (R_1^\beta(\xi), \dots, R_{d-1}^\beta(\xi)).$$

Since we can choose the standard basis in $\mathbb{E}^d$ in such a way that $e_d \in n_\alpha^\perp \cup n_\beta^\perp$, we can assume that $\xi_d \neq 1$.

Multiplying each vector n_{α_ξ}, n_{β_ξ} by $(1-\xi_d)^{d-2}$, we see that our condition on directions $\xi \in S^{d-1} \setminus (\alpha \cup \beta)$ for which $n_{\alpha_\xi} \cdot n_{\beta_\xi} = 0$ can be re-written as

$$f(\xi) := \sum_{l=1}^{d-1} P_l^{\alpha}(\xi) P_l^{\beta}(\xi) = 0.$$

In other words, the desired set is $D_f = Z_f \cap (S^{d-1} \setminus (\alpha \cup \beta))$, where Z_f is the set of zeros of f on S^{d-1}. By geometric considerations, f is not identically zero (it is enough to look at the directions ξ that are close to the ones that are parallel to *span* (α, β)). If the closure of D_f has a non-empty interior then Z_f has a non-empty interior, i.e., there exists a spherical cap $\mathcal{B}_\epsilon(w) \subset Z_f$ of radius $\epsilon > 0$ centered at $w \in S^{d-1}$. But this is impossible, for replacing ξ_d with $\sqrt{1-\xi_1^2-\cdots-\xi_{d-1}^2}$ in the analytic expression for f, we obtain an analytic function of variables $\xi_1, \ldots, \xi_{d-1}$ in an open disc $\mathcal{B}_\epsilon(w)|w^\perp$ centered at the origin. This contradicts the fact that the set of zeros of an analytic function of several real variables is of Lebesgue measure zero (cf. [O]).

Thus, the closure of D_f has an empty interior, which means that D_f is nowhere dense. The Lemma is proved. □

The following elementary result will be crucial in the Proof of Theorem 2.

Lemma 2.3. *Let U be an open subset of S^{d-1} and let $\{l_i\}_{i=1}^4$ be four lines in $\mathbb{E}^d, d \geq 3$, not passing through the origin, such that for any $\xi \in U$ the subspace $\xi^\perp$ intersects each line at a single point $v_i(\xi)$. Assume also that $|v_1(\xi)v_2(\xi)| = |v_3(\xi)v_4(\xi)|$ for any $\xi \in U$.*

1) If l_1 and l_2 are parallel, then all four lines are parallel. In addition, there exists a translation b, such that $l_3 = l_1 + b$ and $l_4 = l_2 + b$ or $l_3 = l_2 + b$ and $l_4 = l_1 + b$ (the last relation can be re-written as $l_3 = -l_1 + c, l_4 = -l_2 + c$ for $c \in \mathbb{E}^d$; here $-l$ is the reflection of the line l in the origin, i.e., $-l = \{-x : x \in l\}$).

2) If l_1 and l_2 are not parallel, then one of the following holds: $l_1 = \pm l_3, l_2 = \pm l_4$ or $l_1 = \pm l_4, l_2 = \pm l_3$.

3) If l_1 and l_2 are not parallel and $dim(span(l_1 \cup l_2)) = 3$, then one of the following holds: $l_1 = l_3, l_2 = l_4$, or $l_1 = -l_3, l_2 = -l_4$, or $l_1 = l_4, l_2 = l_3$, or $l_1 = -l_4, l_2 = -l_3$.

Proof. We start the proof with some elementary algebraic observations.

Parameterize each line $l_i(t) = b_i + ta_i$, such that $t \in \mathbb{R}$, $|a_i| = 1$ and $b_i \cdot a_i = 0$. Notice, that the choice of the directional vectors a_i is determined up to a sign, and the value of the parameter t for the points of intersection can be found from the condition $\xi \cdot (b_i + ta_i) = 0$, i.e., $v_i(\xi) = b_i - \frac{\xi \cdot b_i}{\xi \cdot a_i} a_i$. Hence, the condition of the lemma can be re-written as

$$(2) \qquad \left| b_1 - \frac{\xi \cdot b_1}{\xi \cdot a_1} a_1 - b_2 + \frac{\xi \cdot b_2}{\xi \cdot a_2} a_2 \right| = \left| b_3 - \frac{\xi \cdot b_3}{\xi \cdot a_3} a_3 - b_4 + \frac{\xi \cdot b_4}{\xi \cdot a_4} a_4 \right|, \quad \forall \xi \in U.$$

In other words,

$$\frac{P_1(\xi)}{Q_1(\xi)} = \frac{P_2(\xi)}{Q_2(\xi)}, \quad \forall \xi \in U, \tag{3}$$

where

$$P_1(\xi) = |(\xi \cdot a_1)(\xi \cdot a_2)(b_1 - b_2) - (\xi \cdot b_1)(\xi \cdot a_2)a_1 + (\xi \cdot a_1)(\xi \cdot b_2)a_2|^2,$$
$$P_2(\xi) = |(\xi \cdot a_3)(\xi \cdot a_4)(b_3 - b_4) - (\xi \cdot b_3)(\xi \cdot a_4)a_3 + (\xi \cdot a_3)(\xi \cdot b_4)a_4|^2,$$
$$Q_1(\xi) = |(\xi \cdot a_1)(\xi \cdot a_2)|^2, \qquad Q_2(\xi) = |(\xi \cdot a_3)(\xi \cdot a_4)|^2.$$

By the direct computations, we have

$$P_1(\xi) = (\xi \cdot a_1)^2(\xi \cdot a_2)^2|b_1 - b_2|^2 + (\xi \cdot b_1)^2(\xi \cdot a_2)^2 + (\xi \cdot a_1)^2(\xi \cdot b_2)^2 +$$
$$+2(\xi \cdot b_1)(\xi \cdot a_2)^2(\xi \cdot a_1)(a_1 \cdot b_2) - 2(\xi \cdot a_1)(\xi \cdot b_2)(\xi \cdot a_2)(\xi \cdot b_1)(a_1 \cdot a_2) +$$
$$+2(\xi \cdot a_1)^2(\xi \cdot a_2)(\xi \cdot b_2)(b_1 \cdot a_2).$$

To proceed, we show at first that the left-hand side of (3) is reducible if and only if $l_1 \parallel l_2$.

Indeed, if $l_1 \parallel l_2$, then $a_1 = \pm a_2$, and the fractions are reducible. Using the fact that $a_1 \cdot b_2 = a_2 \cdot b_1 = 0$, we can re-write the left-hand side of (3) as:

$$\frac{P_1(\xi)}{Q_1(\xi)} = \frac{(\xi \cdot a_1)^2|b_1 - b_2|^2 + (\xi \cdot (b_1 - b_2))^2}{(\xi \cdot a_1)^2}. \tag{4}$$

The fraction in (4) is not reducible, for, otherwise, $a_1 = \pm(b_1 - b_2)$, which is not possible due to $(b_1 - b_2) \perp a_1$.

Now assume that the left-hand side of the equation (3) is reducible by, say, $(\xi \cdot a_1)$. Then the second term of $P_1(\xi)$ is reducible by $(\xi \cdot a_1)$. Since $b_1 \perp a_1$, we obtain that $a_2 = \pm a_1$. Similarly, if the left-hand side of (3) is reducible by $(\xi \cdot a_2)$, then the third term is reducible by $(\xi \cdot a_2)$, which gives $a_2 = \pm a_1$ and $l_1 \parallel l_2$.

If the left-hand side is of (3) is reducible, i.e., looks as (4), then its right-hand side must be reducible as well. To see this, compare all directions $\xi \in S^{d-1}$ for which either of the fractions is not defined. In particular, for all $\xi \in S^{d-1}$ such that $\xi \cdot a_1 = 0$, we must have have $\xi \cdot a_3 = 0$ or $\xi \cdot a_4 = 0$. Without loss of generality, assume that $a_1 = \pm a_3$. Then,

$$(\xi \cdot a_1)^2|b_1 - b_2|^2 + (\xi \cdot (b_1 - b_2))^2 = \frac{P_2(\xi)}{(\xi \cdot a_4)^2}.$$

Since the polynomial on the left-hand side of the previous equality is defined for any $\xi \in S^{d-1}$, the right-hand side must be reducible.

Thus, we obtain that $a_1 = \pm a_2$ implies $a_3 = \pm a_4$, i.e., $l_1 \parallel l_2$ implies $l_3 \parallel l_4$.

If both of the fractions are reducible (i.e., $a_1 = \pm a_2$, $a_3 = \pm a_4$), the equality of denominators implies $a_1 = \pm a_2 = \pm a_3 = \pm a_4$. Using the equality of the numerators, we have

$$(\xi \cdot a_1)^4(|b_1 - b_2|^2 - |b_3 - b_4|^2) + ((\xi \cdot (b_1 - b_2))^2 - (\xi \cdot (b_3 - b_4))^2)(\xi \cdot a_1)^2 = 0. \tag{5}$$

By taking $\xi = a_1$ in (5), we obtain $|b_1 - b_2| = |b_3 - b_4|$. We see now that (5) has reduced to

$$((\xi \cdot (b_1 - b_2))^2 - (\xi \cdot (b_3 - b_4))^2)(\xi \cdot a_1)^2 = 0,$$

or to

$$(\xi \cdot (b_1 - b_2 - (b_3 - b_4)))(\xi \cdot (b_1 - b_2 + b_3 - b_4)) = 0 \qquad \forall \xi \in \mathbb{E}^d.$$

This implies

$$b_1 - b_2 = b_3 - b_4 \quad \text{or} \quad b_1 - b_2 = b_4 - b_3.$$

The first part of the lemma is proved.

We prove 2).

In this case, as we have already seen, the fractions in equation (3) are not reducible. We will consider the first four possibilities, other four options are obtained by changing indices $3 \leftrightarrow 4$. Comparing the denominators in (3), we see that $a_1 = \pm a_3$, $a_2 = \pm a_4$. Writing out the numerators in (3), we have

$$\begin{aligned}(\xi \cdot a_1)^2(\xi \cdot a_2)^2(|b_1 - b_2|^2 - |b_3 - b_4|^2) + (\xi \cdot a_2)^2((\xi \cdot b_1)^2 - (\xi \cdot b_3)^2) + \\ + (\xi \cdot a_1)^2((\xi \cdot b_2)^2 - (\xi \cdot b_4)^2) - \\ - 2(\xi \cdot a_1)(\xi \cdot a_2)^2((a_1 \cdot (b_1 - b_2))(\xi \cdot b_1) - (a_1 \cdot (b_3 - b_4))(\xi \cdot b_3)) + \\ + 2(\xi \cdot a_1)^2(\xi \cdot a_2)((a_2 \cdot (b_1 - b_2))(\xi \cdot b_2) - (a_2 \cdot (b_3 - b_4))(\xi \cdot b_4)) - \\ - 2(\xi \cdot a_1)(\xi \cdot a_2)(a_1 \cdot a_2)((\xi \cdot b_1)(\xi \cdot b_2) - (\xi \cdot b_3)(\xi \cdot b_4)) = 0.\end{aligned}$$

Since the second term must be divisible by $\xi \cdot a_1$ and the third one must be divisible by $\xi \cdot a_2$, we see that $b_1 - b_3 = \lambda_1 a_1$ or $b_1 + b_3 = \mu_1 a_1$ for some λ_1, $\mu_1 \in \mathbb{R}$ (observe that both conditions may not hold simulteneously, for, otherwise, summing them up, we would obtain that a_1 is parallel to b_1) and $b_2 - b_4 = \lambda_2 a_2$ or $b_2 + b_4 = \mu_2 a_2$ for some λ_2, $\mu_2 \in \mathbb{R}$ (again, both conditions may not hold simulteneously, otherwise a_2 would be parallel to b_2). If $b_1 - b_3 = \lambda_1 a_1$, then $l_1 = l_3$, for, the parametric equation of l_3 becomes $l_3(t) = b_1 - \lambda_1 a_1 + t a_1$, $t \in \mathbb{R}$, which is the same as the one of $l_1(s) = b_1 + a_1 s$, $s \in \mathbb{R}$, after taking $t = \lambda_1 + s$. Arguing similarly, we see that $l_1 = \pm l_3$, $l_2 = \pm l_4$. This finishes the proof of 2).

In order to prove 3), it remains to exclude two cases $l_1 = l_3$, $l_2 = -l_4$, and $l_1 = -l_3$, $l_2 = l_4$, provided $\dim(\mathrm{span}(l_1, l_2)) = 3$ (the cases obtained by changing the indices $3 \leftrightarrow 4$ are excluded similarly). We will consider the first case and the exclusion of the second case is similar. To this end, assume that $a_3 = a_1$, $b_3 = b_1$, and $a_2 = a_4$, $b_2 = -b_4$. Then, condition (2) reads as

$$\left|\left(b_1 - \frac{\xi \cdot b_1}{\xi \cdot a_1} a_1\right) - \left(b_2 - \frac{\xi \cdot b_2}{\xi \cdot a_2} a_2\right)\right| = \left|\left(b_1 - \frac{\xi \cdot b_1}{\xi \cdot a_1} a_1\right) + \left(b_2 - \frac{\xi \cdot b_2}{\xi \cdot a_2} a_2\right)\right|, \ \forall \xi \in U.$$

It is satisfied only, provided

$$\left(b_1 - \frac{\xi \cdot b_1}{\xi \cdot a_1} a_1\right) \cdot \left(b_2 - \frac{\xi \cdot b_2}{\xi \cdot a_2} a_2\right) = 0 \quad \forall \xi \in U. \tag{6}$$

(Observe that (6) holds in the case $d \geq 4$, $\dim(\mathrm{span}(l_1, l_2)) = 4$, and $span(a_1, b_1) \perp span(a_2, b_2)$).

We show that (6) leads to a contradiction in the case $\dim(\mathrm{span}(l_1, l_2)) = 3$. We may assume that $d = 3$. We observe that indeed $\mathrm{span}(a_1, b_1) = \mathrm{span}(a_2, b_2)$. Passing to the common denominator in (6) and using analiticity, we have

$$(b_1 \cdot b_2)(\xi \cdot a_1)(\xi \cdot a_2) - (b_1 \cdot a_2)(\xi \cdot a_1)(\xi \cdot b_2) -$$
$$(a_1 \cdot b_2)(\xi \cdot b_1)(\xi \cdot a_2) + (a_1 \cdot a_2)(\xi \cdot b_1)(\xi \cdot b_2) = 0 \qquad \forall \xi \in \mathbb{E}^3.$$

Since $d = 3$ and, by the assumption, $a_i \cdot b_i = 0$one of the values $b_1 \cdot b_2$, $b_1 \cdot a_2$, $a_1 \cdot b_2$, $a_1 \cdot a_2$ is not zero. Hence, the left-hand side of the previous equation is not identically zero. Without loss of generality, $b_1 \cdot b_2 \neq 0$. Then, the expression

$$-(a_1 \cdot b_2)(\xi \cdot b_1)(\xi \cdot a_2) + (a_1 \cdot a_2)(\xi \cdot b_1)(\xi \cdot b_2)$$

is divisible by $\xi \cdot a_1$, and, as a consequence,

$$\xi \cdot ((a_1 \cdot a_2)b_2 - (a_1 \cdot b_2)a_2)$$

is divisible by $\xi \cdot a_1$. This gives

$$(a_1 \cdot a_2)b_2 - (a_1 \cdot b_2)a_2 = \mu a_1 \tag{7}$$

for some $\mu \in \mathbb{R}$.

Observe that $\mu \neq 0$. Indeed, if $\mu = 0$, we have $a_1 \cdot a_2 = 0, a_1 \cdot b_2 = 0$, since $a_2 \perp b_2$. Then (6) implies

$$(b_1 \cdot b_2)(\xi \cdot a_1)(\xi \cdot a_2) - (b_1 \cdot a_2)(\xi \cdot a_1)(\xi \cdot b_2) = 0, \quad \forall \xi \in \mathbb{E}^3,$$

or, equivalently

$$(b_1 \cdot b_2)(\xi \cdot a_2) = (b_1 \cdot a_2)(\xi \cdot b_2), \quad \forall \xi \in \mathbb{E}^3.$$

The last condition is not possible, due to $a_2 \perp b_2$. Thus, $\mu \neq 0$.

Similarly, since

$$(b_1 \cdot b_2)(\xi \cdot a_1)(\xi \cdot a_2) - (a_1 \cdot b_2)(\xi \cdot b_1)(\xi \cdot a_2)$$

is divisible by $\xi \cdot a_2$, the expression

$$(\xi \cdot b_2)((a_1 \cdot a_2)(\xi \cdot b_1) - (b_1 \cdot a_2)(\xi \cdot a_1))$$

is divisible by $\xi \cdot a_2$ as well, and we have

$$(a_1 \cdot a_2)b_1 - (b_1 \cdot a_2)a_1 = \lambda a_2, \tag{8}$$

for some $\lambda \in \mathbb{R}$. Following an argument which is similar to the one above, it can be shown that $\lambda \neq 0$. Notice that, by (7), $a_1 \in \mathrm{span}(b_2, a_2)$; and, by (8), $b_1 \in \mathrm{span}(a_1, a_2)$, which implies that $\mathrm{span}(a_1, b_1) = \mathrm{span}(a_2, b_2)$, due to the fact that a_1 is not parallel to a_2.

Finally, denote

$$A(\xi) = (\xi \cdot a_1)b_1 - (\xi \cdot b_1)a_1, \qquad B(\xi) = (\xi \cdot a_2)b_2 - (\xi \cdot b_2)a_2.$$

We know that for every $\xi \in \mathbb{E}^3$ these vectors belong to the same two-dimensional subspace of $\mathbb{E}^3$. Using (6) and the analyticity of $A(\xi)$ and $B(\xi)$, we see that $A(\xi)\cdot\xi = 0$ and $B(\xi)\cdot\xi = 0$ for all $\xi \in \mathbb{E}^3$, i.e., $A(\xi)$ and $B(\xi)$ belong to $\xi^\perp$. Moreover, due to (6) they are orthogonal to each other. By taking $\xi \in \mathrm{span}(a_1, b_1)$, we obtain two vectors that are orthogonal and parallel to each other at the same time. Hence, at least one of them is zero, and we obtain a contradiction. The lemma is proved. □

Lemma 2.4. *Let l_p, l_q, l_r be three distinct lines in $\mathbb{E}^d$ and $\xi \in S^{d-1}$, such that ξ is not orthogonal to l_i for $i = p, q, r$. Denote $v_i(\xi) = l_i \cap \xi^\perp$. Then the set of directions $\xi \in S^{d-1}$, such that non-zero vectors $v_q(\xi) - v_p(\xi)$ and $v_r(\xi) - v_p(\xi)$ are orthogonal, is a nowhere dense subset $Y_{pqr} \subset S^{d-1}$.*

Proof. Let $l_i(t) = b_i + ta_i$ $(t \in \mathbb{R}; i = p, q, r)$, then $v_i(\xi) = b_i - \frac{b_i\cdot\xi}{a_i\cdot\xi}a_i$. Consider a function $f : S^{d-1} \to \mathbb{R}$, defined as following

$$f(\xi) = \left(b_q - \frac{b_q\cdot\xi}{a_q\cdot\xi}a_q - b_p + \frac{b_p\cdot\xi}{a_p\cdot\xi}a_p\right)\cdot\left(b_r - \frac{b_r\cdot\xi}{a_r\cdot\xi}a_r - b_p + \frac{b_p\cdot\xi}{a_p\cdot\xi}a_p\right).$$

The condition of orthogonality of the given vectors in terms of ξ is $f(\xi) = 0$. By geometrical considerations, $f \not\equiv 0$ on S^{d-1}. Since $a_i \cdot \xi \neq 0$, then, multiplying both sides by $(a_p\cdot\xi)(a_q\cdot\xi)(a_r\cdot\xi)$, the condition is equivalent to a polynomial equation of third degree in coordinates of ξ.

Using the argument which is similar to the one in Lemma 2.2, one can show that the set of zeroes $Z_f \subset S^{d-1}$ of f is nowhere dense on S^{d-1}.

□

We will use the well-known Minkowski theorem (see, for example, [K], Theorem 9, p. 282).

Theorem 3. *Suppose that K and L are polytopes in $\mathbb{E}^d, d \geq 2$, and suppose also that the facet unit normals $n^1, \dots, n^l$ and corresponding facet areas $c_1, \dots, c_l$ coincide, then K coincides with L up to a translation.*

Our last auxiliary result in this section is

Theorem 4 (see [M], Theorem 1.3)**.** *Let $2 \leq j \leq d-1$ and let f and g be two continuous real-valued functions on S^{d-1}. Assume that for any j-dimensional subspace α and some vector $a_\alpha \in \alpha$, the restrictions of f and g onto $S^{d-1}\cap\alpha$ satisfy $f(-u) + a_\alpha\cdot u = g(u) \quad \forall u \in \alpha\cap S^{d-1}$ or $f(u) + a_\alpha\cdot u = g(u) \quad \forall u \in \alpha\cap S^{d-1}$. Then there exists $b \in \mathbb{E}^d$ such that $g(u) = f(u) + b\cdot u$ $\forall u \in S^{d-1}$ or $g(u) = f(-u) + b\cdot u$ $\forall u \in S^{d-1}$.*

We remark also that if all a_α are zero, then $b = 0$ (see also Lemma 1 from [R2], page 3431).

3. Proof of Theorem 1

We start with the case $2 \le k = d-1$.

3.1. **Main idea.** We will show that for all directions $\xi \in S^{d-1}$, the projections of polytopes onto $\xi^\perp$ coincide up to a translation and a reflection in the origin. This will be achieved in three steps that can be briefly sketched as follows.

Step 1. Let $Y \subset S^{d-1}$ be a closed set of directions ξ that are parallel to facets of P or Q, and let U_1 be any connected component of $U = S^{d-1} \setminus Y$. We will prove that, given any two vectors ξ_1 and ξ_2 in U_1, we have $\partial_{\xi_1} P = \partial_{\xi_2} P$ and $\partial_{\xi_1} Q = \partial_{\xi_2} Q$; see Lemma 3.1.

Step 2. We will prove that the edges of the shadow boundaries $\partial_\xi P$ and $\partial_\xi Q$, $\xi \in U$, are in a bijective correspondence. We will apply Lemma 2.1 to all corresponding edges of $\partial_\xi P$ and $\partial_\xi Q$ to conclude that they are parallel and have equal length.

Step 3. We will show that the corresponding facets of projections P_ξ and $Q_\xi = T_\xi(P_\xi)$, $\xi \in U$, are pairwise parallel. We will apply Minkowski's Theorem to conclude that P_ξ and Q_ξ, $\xi \in U$, coincide up to a translation and a reflection in the origin. Then, we will use the density argument to conclude that the last statement holds for all directions $\xi \in S^{d-1}$.

Finally, we will apply Theorem 4 with f and g being the support functions of polytopes.

3.2. **Proof.** *Step 1.* Fix any facet F of P or Q with an outer unit normal η. Any direction $\xi \in S^{d-1}$ which is parallel to F belongs to $\eta^\perp$. Since polytopes have a finite number of facets, the set Y of all such directions ξ is a union of a finite number of great subspheres of S^{d-1}. Denote $U = S^{d-1} \setminus Y$. Since Y is closed, U is open.

Fix any $\xi \in U$. Any vertex $v_\xi \in P_\xi$ is a projection of some vertex (not an edge), otherwise ξ is parallel to some facet. The same holds for edges of P_ξ, i.e., a pre-image of any edge of P_ξ is an edge of P, otherwise ξ is parallel to some 2-dimensional face of P.

Let U_1 be any non-empty linearly connected component of U. Observe that it is contained in an open hemisphere S^{d-1}_ζ for some $\zeta \in S^{d-1}$. Note also that U_1 is geodesically convex, i.e., for any two directions ξ_1 and ξ_2 in U_1 there exists an arc $[\xi_1, \xi_2]$ of a great circle of S^{d-1} such that $[\xi_1, \xi_2] \subset U_1$. Indeed, since the boundary ∂U_1 of U_1 is a finite union of closed pieces of great subspheres of S^{d-1}, $\partial U_1 = \bigcup\limits_{k=1}^{j} S_k$, we see that $U_1 = S^{d-1} \cap \mathcal{H}$, where $\mathcal{H}$ is a finite convex intersection of j half-spaces defined by hyperplanes, passing through the origin and containing S_k. Since $\mathcal{H}$ is convex, the interval $\xi_1\xi_2$ connecting ξ_1 and ξ_2 belongs to $\mathcal{H}$. The projection of this interval onto S^{d-1} is an arc $[\xi_1, \xi_2]$ of the unit circle, which is contained in U_1.

We have

Lemma 3.1. *For any $\xi_1, \xi_2 \in U_1$ the shadow boundaries coincide, $\partial_{\xi_1} P = \partial_{\xi_2} P$.*

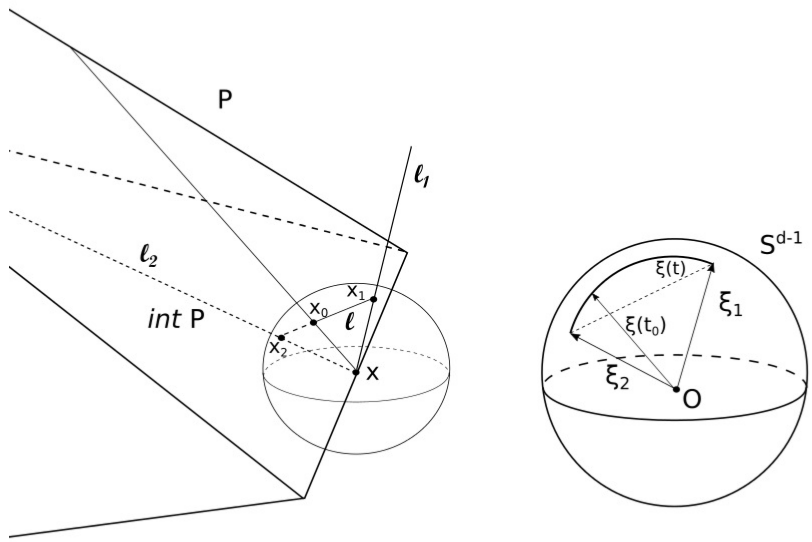

FIGURE 1. Preserving of shadow boundaries.

Proof. Assume the opposite, there exist two distinct vectors $\xi_1, \xi_2 \in U_1$, such that $\partial_{\xi_1} P \neq \partial_{\xi_2} P$. Then there exists $x \in \partial_{\xi_1} P$, but $x \notin \partial_{\xi_2} P$, and we may construct two half-lines $l_1(t_1) = x + t_1\xi_1, t_1 \geq 0$, and $l_2(t_2) = x + t_2\xi_2, t_2 \geq 0$ (or $t_2 \leq 0$, if necessary) to obtain

$$l_1 \cap intP = \emptyset \quad \text{and} \quad l_2 \cap intP \neq \emptyset.$$

Take a small enough $\varepsilon > 0$, such that the ball $B(x, \varepsilon)$ intersects only faces of P that contain x (see Figure 1). Choose $x_2 \in l_2 \cap intP \cap B(x, \varepsilon)$ and $x_1 \in l_1 \cap B(x, \varepsilon), x_1 \neq x$. Notice that $x_1 \notin \partial P$, otherwise $\frac{x_1 - x}{|x_1 - x|} \notin U$, which is not possible since ξ_1 is parallel to $\frac{x_1 - x}{|x_1 - x|}$.

We project the interval $x_1x_2 = \{tx_1 + (1-t)x_2, t \in [0,1]\}$ onto S^{d-1} and obtain an arc $\xi(t) = \frac{tx_1 + (1-t)x_2 - x}{|tx_1 + (1-t)x_2 - x|}$ of the unit circle $span\{\xi_1, \xi_2\} \cap S^{d-1}$. On the other hand, there exists a point x_0, such that $x_0 = t_0x_1 + (1-t_0)x_2 \in \alpha$ for some $t_0 \in (0,1)$, where $\alpha \subset \partial P$ is a facet of P containing x_0. We have $\xi(t_0) \notin U_1$, which contradicts the fact that U_1 is geodesically convex. □

Step 2. Our next goal is to show that the edges of shadow boundaries $\partial_\xi P$ and $\partial_\xi Q$, $\xi \in U_1$, are in a bijective correspondence. Moreover, we will prove that the corresponding edges are parallel and have equal length.

Assume that for a fixed $\xi \in U_1$ the projection P_ξ has k vertices $\{(v_1)_\xi, (v_2)_\xi, \dots, (v_k)_\xi\}$ which are the projections of k vertices $\{v_1, v_2, \dots, v_k\}$ of P. In each hyperplane $\xi^\perp$ we consider a rigid motion T_ξ, such that $T_\xi(P_\xi) = Q_\xi$ (if there are several such T_ξ, take any). It is clear that every vertex $(v_i)_\xi \in P_\xi$ is mapped onto some vertex

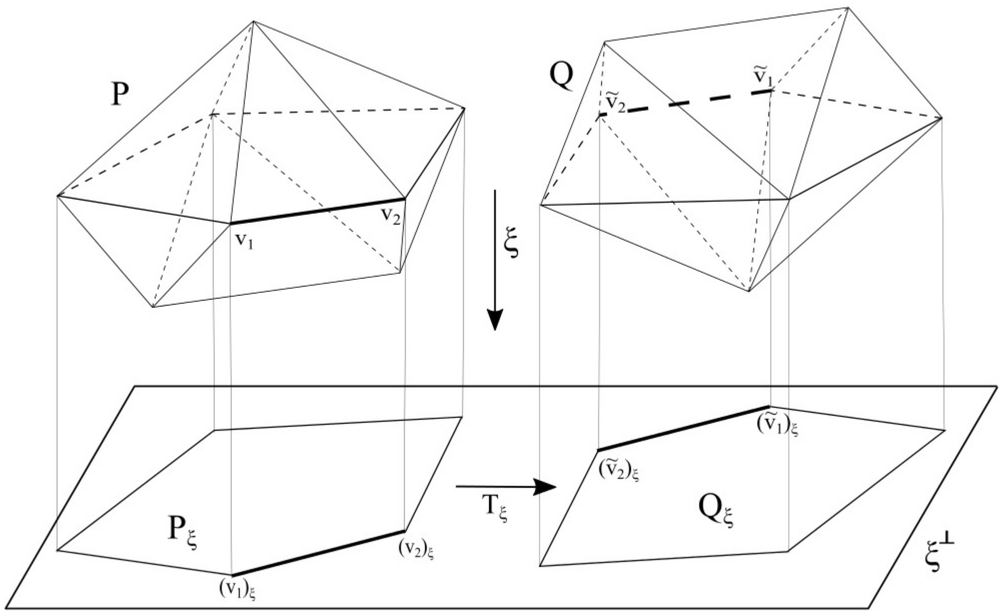

FIGURE 2. Correspondence of vertices through projections

$(\tilde{v}_j)_\xi = T_\xi((v_i)_\xi) \in Q_\xi$ and every edge of P_ξ is mapped onto some edge of Q_ξ (notice also that the pre-image of any vertex of Q_ξ is a vertex of Q and the pre-image of any edge of Q_ξ is an edge of Q). This implies that for any $\xi \in U_1$ we obtain a bijective correspondence f_ξ between the set of all vertices $\{v_1, v_2, ..., v_k\}$ of the shadow boundary $\partial_\xi P$ and the set of all vertices $\{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_k\}$ of the shadow boundary $\partial_\xi Q$.

Take a closed spherical cap with a non-empty interior $W \subset U_1$. For any $\xi \in W$ we have at least one T_ξ satisfying $T_\xi(P_\xi) = Q_\xi$. Hence, for any $\xi \in W$, we have at least one map $f_\xi : \{v_1, v_2, ..., v_k\} \to \{\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_k\}$, such that $f_\xi(v_i) = \tilde{v}_{\sigma_\xi(i)}$, and σ_ξ is a permutation of the set $\{1, 2, ..., k\}$ satisfying $(\tilde{v}_{\sigma_\xi(i)})_\xi = T_\xi((v_i)_\xi)$ (see Figure 2). The set of all such possible maps $\{f_\xi\}_{\xi \in W}$ is finite. We have

$$W = \bigcup_{\sigma \in \mathcal{P}_k} V_\sigma, \qquad V_\sigma = \{\xi \in W : \exists f_\xi \text{ such that } f_\xi(v_i) = \tilde{v}_{\sigma(i)} \quad \forall i = 1, \dots, k\},$$

where $\mathcal{P}_k$ is the set of all permutations of $\{1, 2, ..., k\}$.

Observe that each V_σ is a closed set (it might be empty). Indeed, let $(\xi_k)_{k=1}^\infty$ be a convergent sequence of points of a non-empty V_σ, and let $\lim\limits_{k\to\infty} \xi_k = \xi$. We have $T_{\xi_k}((v_i)_{\xi_k}) = (\tilde{v}_{\sigma(i)})_{\xi_k}$, i.e.,

$$T_{\xi_k}(v_i - (v_i \cdot \xi_k)\xi_k) = \tilde{v}_{\sigma(i)} - (\tilde{v}_{\sigma(i)} \cdot \xi_k)\xi_k, \tag{9}$$

where σ is independent of ξ_k. For each $\xi \in W$ extend the operator T_ξ acting from $\xi^\perp$ to $\xi^\perp$ to the operator $\mathbb{T}_\xi$ acting from $\mathbb{E}^d$ to $\mathbb{E}^d$ as $\mathbb{T}_\xi(a) = \Phi_\xi(a) + b_\xi$. Here, $\Phi_\xi \in O(d)$

is defined as $\Phi_\xi|_{\xi^\perp} = \varphi_\xi$, $\Phi_\xi(\xi) = \xi$, where $T_\xi(a_\xi) = \varphi_\xi(a_\xi) + b_\xi$. Equation (9) in terms of $\mathbb{T}_{\xi_k}$ can be re-written as

$$\mathbb{T}_{\xi_k}(v_i - (v_i \cdot \xi_k)\xi_k) = \Phi_{\xi_k}(v_i) - (v_i \cdot \xi_k)\xi_k + b_{\xi_k} = \tilde{v}_{\sigma(i)} - (\tilde{v}_{\sigma(i)} \cdot \xi_k)\xi_k.$$

Without loss of generality, both polytopes P and Q are located inside a large ball. Hence, the entries $\{a^k_{ij}\}_{k\in\mathbb{N}}, i, j = 1, 2, ..., d$, of the matrix corresponding to transformations Φ_{ξ_k} and the coordinates $b^k_j, j = 1, \dots, d$ of vector b_{ξ_k} are bounded functions of ξ_k. By compactness, we can assume that all $\{a^k_{ij}\}_{k\in\mathbb{N}}$, and $\{b^k\}_{k\in\mathbb{N}}$ are convergent to the corresponding entries of $\tilde{\Phi}_\xi = \lim_{k\to\infty} \Phi_{\xi_k}$ and $\tilde{b}_\xi = \lim_{k\to\infty} b_{\xi_k}$ respectively. This yields

$$\tilde{\mathbb{T}}_\xi\left((v_i)_\xi\right) = \tilde{\Phi}_\xi(v_i - (v_i \cdot \xi)\xi) + \tilde{b}_\xi = T_\xi((v_i)_\xi) = \tilde{v}_{\sigma(i)} - (\tilde{v}_{\sigma(i)} \cdot \xi)\xi = (\tilde{v}_{\sigma(i)})_\xi,$$

where $\tilde{\mathbb{T}}_\xi = \lim_{k\to\infty} \mathbb{T}_{\xi_k}$. In other words, there exists $\tilde{T}_\xi$, such that the corresponding $\tilde{f}_\xi$ satisfies $\tilde{f}_\xi(v_i) = \tilde{v}_{\sigma(i)}, \forall i = 1, \dots, k$. This means that $\xi \in V_\sigma$ and V_σ is a closed set.

By the Baire category Theorem (see, for example, [R], pages 42-43) there exists a permutation σ_o such that the interior U_o of V_{σ_o} is non-empty. (This could be also easily seen as follows. Enumerate $\mathcal{P}_k$, and take the first set V_{σ_1}. If its interior is empty, it is nowhere dense, and for every open spherical cap $\mathcal{B}_1$ in W there exists a smaller cap $\mathcal{B}_2$ that is free of points of V_{σ_1}, $\mathcal{B}_2 \cap V_{\sigma_1} = \emptyset$. Repeat the procedure with V_{σ_2} and $\mathcal{B}_2$ instead of V_{σ_1} and $\mathcal{B}_1$. After finitely many steps, unless we meet some V_σ with a non-empty interior, we will obtain a spherical cap that does not intersect W, which is impossible).

Observe that if v_i and v_j are connected by an edge $v_iv_j \subset P$, such that $(v_i)_\xi(v_j)_\xi$ is an edge of P_ξ, $\xi \in U_o$, then $\tilde{v}_{\sigma(i)}\tilde{v}_{\sigma(j)}$ is an edge of Q. Now, we can apply Lemma 2.1 to all such pairs of edges $v_iv_j \in \partial_\xi P$ and $\tilde{v}_{\sigma(i)}\tilde{v}_{\sigma(j)} \in \partial_\xi Q$ for any $\xi \in U_o \subset S^{d-1}$. We see that these edges are parallel and have equal length. Thus, all corresponding edges belonging to the shadow boundaries $\partial_\xi P$ and $\partial_\xi Q$, $\xi \in U_o$, are parallel and have equal lengths. Applying Lemma 3.1, we conclude that the last statement holds for U_1 instead of U_o.

Step 3. We will show that the projections of both polytopes in the directions of U_1 coincide up to a translation and a reflection in the origin. To do this, we will use Minkowski's Theorem about uniqueness (up to a translation) of polytopes with parallel facets having the same volume. Our polytopes will be P_ξ and Q_ξ, $\xi \in U_1$.

Fix any $\xi \in U_1$. Then P_ξ and Q_ξ are two $(d-1)$-dimensional polytopes in $\xi^\perp$, such that $T_\xi(P_\xi) = Q_\xi$. Since the map T_ξ is a bijection between the sets of all facets of P_ξ and Q_ξ, for any facet α of P_ξ there exists a unique facet $\tilde{\alpha}$ of Q_ξ such that $T_\xi(\alpha) = \tilde{\alpha}$.

We will show at first that α is parallel to $\tilde{\alpha}$. Consider the $(d-2)$-dimensional affine subspaces Π, $\tilde{\Pi}$ of $\xi^\perp$ such that $\alpha \subset \Pi$ and $\tilde{\alpha} \subset \tilde{\Pi}$. We claim that $\tilde{\Pi}$ is parallel to Π. In other words, the outer unit normals $n_\alpha, n_{\tilde{\alpha}} \in \xi^\perp$ of Π and $\tilde{\Pi}$ satisfy $n_{\tilde{\alpha}} = \pm n_\alpha$.

To prove the claim, observe that there exist $d-2$ linearly independent non-zero directional vectors $\{(a_1)_\xi, \dots, (a_{d-2})_\xi\}$ of edges of α having a common vertex, such that $span\{(a_1)_\xi, \dots, (a_{d-2})_\xi\}$ is parallel to Π (here, a directional vector of an edge is any non-zero vector parallel to the edge). The directional vectors of edges $\{(a_1)_\xi, \dots, (a_{d-2})_\xi\}$ are the projections of directional vectors of the edges $\{a_1, \dots, a_{d-2}\} \subset \partial_\xi P$. The directional vectors $\{\tilde{a}_1, \dots, \tilde{a}_{d-2}\} \subset \partial_\xi Q$ of the corresponding edges are parallel, i.e., a_i is parallel to $\tilde{a}_i$ for $i = 1, \dots, d-2$. The same holds true for their projections onto $\xi^\perp$, $(a_i)_\xi$ is parallel to $(\tilde{a}_i)_\xi$ for $i = 1, \dots, d-2$. We conclude that for the $(d-2)$-dimensional affine subspace $\tilde{\Pi}$ containing $\tilde{\alpha}$ we have $T_\xi(\Pi) = \tilde{\Pi}$ and $\tilde{\Pi}$ is parallel to $span\{(\tilde{a}_1)_\xi, \dots, (\tilde{a}_{d-2})_\xi\} = span\{(a_1)_\xi, \dots, (a_{d-2})_\xi\}$. The claim is proved.

Since T_ξ is an isometry, we have $\mathrm{vol}_{d-2}(\alpha) = \mathrm{vol}_{d-2}(\tilde{\alpha})$. Now assume that both polytopes P_ξ and Q_ξ have l facets $\{\alpha_i\}_{i=1}^l \subset P_\xi$ and $\{\tilde{\alpha}_i\}_{i=1}^l \subset Q_\xi$ with corresponding outer normals $\{n_i\}_{i=1}^l$ and $\{\tilde{n}_i\}_{i=1}^l$. We will show that up to a nowehere dense subset of directions ξ (defined below) the projections P_ξ and Q_ξ are translates of each other (up to a reflection in the origin). To be able to apply Minkowski's Theorem to P_ξ and Q_ξ (or $-P_\xi$ and Q_ξ) we have to show that $n_i = \tilde{n}_i$ for all $i = 1, \dots, l$, or $-n_i = \tilde{n}_i$ for all $i = 1, \dots, l$.

Assume that the last statement is not true, and let $\alpha_i, \alpha_j \subset P_\xi$ and $\tilde{\alpha}_i, \tilde{\alpha}_j \subset Q_\xi$, be two pairs of facets such that $\alpha_i \cap \alpha_j \neq \emptyset$, $T_\xi(\alpha_i) = \tilde{\alpha}_i$, $T_\xi(\alpha_j) = \tilde{\alpha}_j$, but $\tilde{n}_i = n_i, \tilde{n}_j = -n_j$, (since T_ξ is an isometry, $\tilde{\alpha}_i \cap \tilde{\alpha}_j \neq \emptyset$). Consider two cases, $n_i \cdot n_j \neq 0$ and $n_i \cdot n_j = 0$. The first case $n_i \cdot n_j \neq 0$ is impossible, for

$$n_i \cdot (-n_j) = \tilde{n}_i \cdot \tilde{n}_j = \varphi_\xi(n_i) \cdot \varphi_\xi(n_j) = \varphi_\xi^T \varphi_\xi(n_i) \cdot n_j = n_i \cdot n_j,$$

yields $n_i \cdot n_j = 0$. Here φ_ξ^T stands for the transpose operator of φ_ξ, and we used the fact that $\varphi_\xi^T \varphi_\xi = I$ due to $\varphi_\xi \in O(d-1, \xi^\perp)$.

To exclude the case $n_i \cdot n_j = 0$, we recall that the pre-image of any facet of P_ξ or Q_ξ is an $(d-2)$-dimensional face of P or Q respectively. We apply Lemma 2.2 to the pairs of subspaces that are parallel to all pairs of $(d-2)$-dimensional faces of P and Q. We obtain a closed nowhere dense subset $Y_1 \subset U$, such that the normals of the corresponding facets of P_ξ and Q_ξ, $\xi \in Y_1$, are orthogonal. Now we may repeat our considerations for the case $n_i \cdot n_j \neq 0$ on $U_1 \setminus Y_1$ instead of U_1. We obtain that for every $\xi \in U_1 \setminus Y_1$ only one of the choices $n_i = \tilde{n}_i$ for all $i = 1, \dots, l$, or $-n_i = \tilde{n}_i$ for all $i = 1, \dots, l$, holds.

We conclude that for all $\xi \in U_1 \setminus Y_1$, we have $n_i = \tilde{n}_i$ for all $i = 1, \dots, l$, or $-n_i = \tilde{n}_i$ for all $i = 1, \dots, l$. Also, recall that $c_i = \tilde{c}_i$ for all $i = 1, \dots, l$, where $c_i = \mathrm{vol}_{d-2}(\alpha_i)$, $\tilde{c}_i = \mathrm{vol}_{d-2}(\tilde{\alpha}_i)$; now we may apply Theorem 3 to P_ξ and Q_ξ or to $-P_\xi$ and Q_ξ.

Thus, for every $\xi \in U_1 \setminus Y_1$ there exists a vector $b_\xi \in \xi^\perp$, such that $Q_\xi = P_\xi + b_\xi$ or $Q_\xi = -P_\xi + b_\xi$. We can repeat the above argument for every connected component of U. We see that for every $\xi \in U \setminus Y_1$ the supporting functions h_P and h_Q satisfy

$$h_Q(u) = h_P(u) + b_\xi \cdot u \quad \text{for any} \quad u \in \xi^\perp, \tag{10}$$

or

$$h_Q(u) = h_P(-u) + b_\xi \cdot u \quad \text{for any} \quad u \in \xi^\perp. \tag{11}$$

Here we used the fact that $h_{P_\xi}(u) = h_P(u)\ \forall u \in \xi^\perp$, see ([Ga], (0.21), p.17).

It is not difficult to see that one of the above equalities (or both) hold for every $\xi \in S^{d-1}$. Indeed, since $Y \cup Y_1$ is nowhere dense on S^{d-1}, for any open neighborhood $V_\xi \subset S^{d-1}$ of any $\xi \in Y \cup Y_1$ we have $V_\xi \cap U \neq \emptyset$. Hence, there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset U$, such that $\lim_{k\to\infty} \xi_k = \xi$. By taking a convergent subsequence if necessary, we conclude that (the argument is very similar to the one in the proof of Lemma 6, p. 3435, [R2]) there exists a limit $\lim\limits_{k\to\infty} b_{\xi_k} = b_\xi \in \xi^\perp$ for $\xi \in Y \cup Y_1$, such that (10) or (and) (11) holds.

To finish the proof in the case $k = d-1$, we apply Theorem 4 with $j = k$, and $f = h_P$, $g = h_Q$. We obtain that there exists $b \in \mathbb{E}^d$ such that $h_Q(u) = h_P(u) + b \cdot u$ for all $u \in \mathbb{E}^d$, or $h_Q(u) = h_P(-u) + b \cdot u$ for all $u \in \mathbb{E}^d$. Using the well-known properties of the support functions ([Ga], pp. 16-18) we obtain the desired result in the case $k = d-1$.

3.3. **Proof of Theorem 1 in the case** $2 \leq k < d-1$**.** We use induction on k. Let H be any $(k+1)$-dimensional subspace of $\mathbb{E}^d$. We apply the result for $d = k+1$ to the bodies $P|H$ and $Q|H$ and their projections $(P|H)|J$ and $(Q|H)|J$ for all k-dimensional subspaces $J \subset H$. We obtain that $P|H = Q|H + b_H$ or $P|H = -Q|H + b_H$ for all H. We proceed and the result follows after finitely many steps. The proof of Theorem 1 is completed.

4. Proof of Theorem 2

We start with the case $2 \leq k = d-1$.

4.1. **Main idea.** We will show that for all directions $\xi \in S^{d-1}$, the sections of polytopes by $\xi^\perp$ coincide up to a reflection in the origin. This will be achieved in four steps, which can be briefly described as follows.

Step 1. Let $Y \subset S^{d-1}$ be a closed set of directions ξ for which $\xi^\perp$ contains any of the vertices of P or Q, and let U_1 be any connected component of $U = S^{d-1} \setminus Y$. We will prove that given any two directions ξ_1 and ξ_2 in U_1, the subspaces $\xi_1^\perp$ and $\xi_2^\perp$ intersect the same set of edges of P (and Q); see Lemma 4.1.

Denote by $E = E(U_1)$ and $\tilde{E} = \tilde{E}(U_1)$ the sets of edges of P and Q that are intersected by $\xi^\perp$, $\xi \in U_1$. Denote also by $L = L(U_1)$ and $\tilde{L} = \tilde{L}(U_1)$ the sets of lines containing the intersected edges from E and $\tilde{E}$.

Step 2. We will prove that there exists an open non-empty subset U_o of U_1 such that for all directions $\xi \in U_o$ all rigid motions T_ξ "look alike". More precisely, for all $\xi \in U_o$ and for any $l_i \in L$ the vertices $v_i(\xi) = \xi^\perp \cap l_i$ of sections $P \cap \xi^\perp$ are all mapped into the vertices of $Q \cap \xi^\perp$ of the form $\tilde{v}_{j(i)}(\xi) = \xi^\perp \cap \tilde{l}_{j(i)}$, where the line $\tilde{l}_{j(i)} \in \tilde{L}$ is fixed and $j = j(i)$ is independent of ξ.

Step 3. Using Lemma 2.3 we will show that the corresponding lines in L and $\tilde{L}$ coincide up to a reflection in the origin.

Step 4. We will apply Theorem 4 with $j = d-1$, $a_\alpha = 0$, $b = 0$, and f and g being the radial functions of polytopes.

4.2. **Proof.** *Step 1.* Consider any vertex v of P or Q. If a hyperplane $\xi^\perp$ contains v then $\xi \cdot v = 0$. Hence, the set of all such directions ξ is a sub-sphere $v^\perp \cap S^{d-1}$. Since both polytopes have a finite number of vertices, the union Y of all such sub-spheres is closed, hence its complement $U = S^{d-1} \setminus Y$ is open in S^{d-1}.

Fix any connected component U_1 of U.

Lemma 4.1. *For any two distinct vectors $\xi_1, \xi_2 \in U_1$, the hyperplanes $\xi_1^\perp$ and $\xi_2^\perp$ intersect the same set of edges of P.*

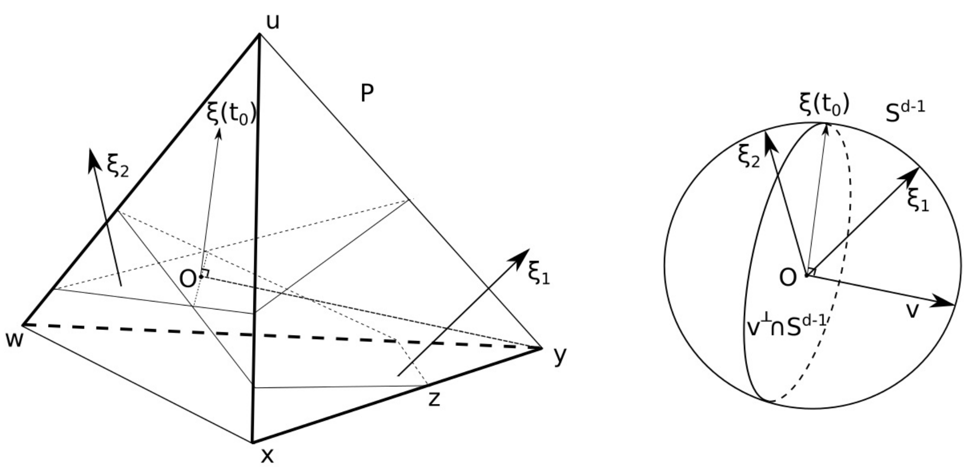

FIGURE 3. The sets $A_1 = \{xy, yw, wu, ux\}$, $A_2 = \{uw, ux, uy\}$.

Proof. Let A_i be the set of all interiors of all edges of P that have non-empty intersections with $\xi_i^\perp$, $i = 1, 2$. We claim that $A_1 = A_2$.

Assume that $A_1 \neq A_2$ (see Figure 3). Then, there exists an edge xy such that $xy \cap \xi_2^\perp = \emptyset$, $xy \cap \xi_1^\perp = z$, for some point $z \in xy$. Assume also that xy lies on a line l, $l(s) = b + sa$, $a, b \in \mathbb{E}^d$, $s \in \mathbb{R}$. To prove the claim consider a continuous path $\xi(t)$ along the shortest arc $[\xi_1\xi_2] \subset U_1$, such that $\xi_1 = \xi(0)$, $\xi_2 = \xi(1)$ (since U_1 is geodesically convex, $[\xi_1\xi_2] \in U_1$).

For any $t \in (0,1)$, the intersection of $\xi(t)^{\perp}$ with l, if exists, can be found as $r(t) = b - \frac{\xi(t)\cdot b}{\xi(t)\cdot a}a$. In particular, $z = r(0) = b - \frac{\xi(0)\cdot b}{\xi(0)\cdot a}a \in xy$ and $r(1) = b - \frac{\xi(1)\cdot b}{\xi(1)\cdot a}a \notin xy$.

Assume that there exists $t_1 \in (0,1]$ such that $\xi(t_1)^{\perp}$ is parallel to l, i.e., $\xi(t_1)\cdot a = 0$ (note that there is at most one such value t_1, since $[\xi_1\xi_2]$ and $a^{\perp}$ intersect at most at one point). In this case function $d(t) = |r(t) - r(0)|$ has a vertical asymptote at t_1, which implies that there exists a small enough $\delta > 0$, such that $d(t_1 - \delta) > |x - y|$, i.e., $r(t_1 - \delta) \notin xy$. Notice that $\xi(t_1 - \delta)$ is not orthogonal to a, so line l is not parallel to hyperplane $\xi(t_1 - \delta)^{\perp}$, i.e., $l \cap \xi(t_1 - \delta)^{\perp} \neq \emptyset$.

Since $r(t)$ is continuous on $[0, t_1 - \delta]$, there exists $t_0 \in (0, t_1 - \delta)$, such that $r(t_0) = x$ (or y). This gives $\xi(t_0) \in Y$, which leads to a contradiction.

If $\forall t \in (0,1]$, $\xi(t)\cdot a \neq 0$, repeat the previous argument with $\delta = 0$. $\square$

Step 2. For any $\xi \in U_1$ the vertices v_i of $P \cap \xi^{\perp}$ are the intersections of $\xi^{\perp}$ with lines $l_i \in L$ and the vertices $\tilde{v}_i$ of $Q \cap \xi^{\perp}$ are the intersections of $\xi^{\perp}$ with lines $\tilde{l}_i \in \tilde{L}$. Since a rigid motion T_ξ maps the vertices of $P \cap \xi^{\perp}$ onto the vertices of $Q \cap \xi^{\perp}$, there exists a one to one correspondence between these sets of vertices. Hence, there also exists a one to one correspondence between the lines in L and $\tilde{L}$ (see Figure 4).

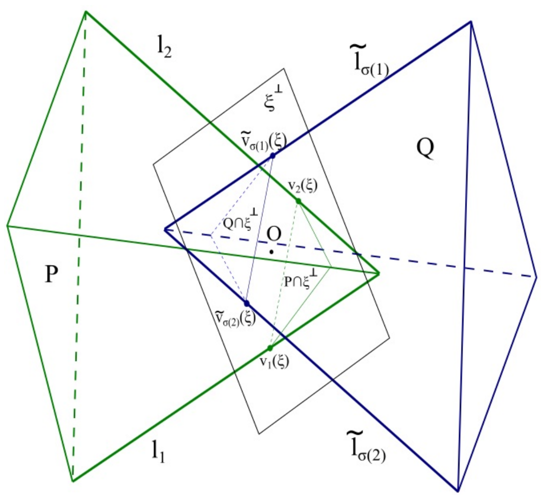

FIGURE 4. Correspondence of lines containing edges through the sections.

Let $L = \{l_i\}_{i=1}^k$ and $\tilde{L} = \{\tilde{l}_i\}_{i=1}^k$. Then for every $\xi \in U_1$, there exists at least one rigid motion T_ξ that maps any vertex $v_i(\xi) = \xi^{\perp} \cap l_i$ of $P \cap \xi^{\perp}$ into the vertex $\tilde{v}_j(\xi) = \xi^{\perp} \cap \tilde{l}_j$ of $Q \cap \xi^{\perp}$, $j = j(i,\xi)$. Hence, there is a permutation $\sigma_\xi \in \mathcal{P}_k$ of the set $\{1,\dots,k\}$ and the map $f_\xi : L \to \tilde{L}$ such that $f_\xi(l_i) = \tilde{l}_{\sigma_\xi(i)}$, $\sigma_\xi(i) = j(i,\xi)$.

We claim that there exists an open non-empty subset U_o of U_1 and a fixed permutation $\sigma = \sigma(U_o) \in \mathcal{P}_k$ such that

$$f_\xi(l_i) = \tilde{l}_{\sigma(i)} \qquad \forall \xi \in U_o, \qquad \forall i = 1,\dots,k. \tag{12}$$

Take a closed spherical cap with a non-empty interior $W \subset U_1$. The set of all possible maps $\{f_\xi\}_{\{\xi \in W\}}$ is finite. Hence,

$$W = \bigcup_{\sigma \in \mathcal{P}_k} V_\sigma, \qquad V_\sigma = \{\xi \in W : \exists f_\xi \text{ such that } f_\xi(l_i) = \tilde{l}_{\sigma(i)} \quad \forall i = 1, \dots, k\}.$$

Observe each V_σ is a closed set (it might be empty). Indeed, let $(\xi_k)_{k=1}^\infty$ be a convergent sequence of points of a non-empty V_σ, and let $\lim_{k\to\infty} \xi_k = \xi$. Then $f_{\xi_k}(l_i) = \tilde{l}_{\sigma(i)}$. In other words, for the corresponding vertices $v_i(\xi_k)$ and $\tilde{v}_{\sigma(i)}(\xi_k)$ we have

$$T_{\xi_k}(v_i(\xi_k)) = \tilde{v}_{\sigma(i)}(\xi_k), \qquad v_i(\xi_k) = l_i \cap \xi_k^\perp, \quad \tilde{v}_{\sigma(i)}(\xi_k) = \tilde{l}_{\sigma(i)} \cap \xi_k^\perp, \tag{13}$$

where σ is independent of ξ_k. Arguing as in the case of projections (see *Step 2)* in the proof of Theorem 1), we extend the operator $T_\xi : \xi^\perp \to \xi^\perp$, $\xi \in W$, to the one acting on the whole space $\mathbb{T}_\xi : \mathbb{E}^d \to \mathbb{E}^d$, $\xi \in W$, by the formula $\mathbb{T}_\xi(a) = \Phi_\xi(a) + b_\xi$. Here, $\Phi_\xi \in O(d)$ is defined as $\Phi_\xi|_{\xi^\perp} = \varphi_\xi$, $\Phi_\xi(\xi) = \xi$, where $T_\xi(a_\xi) = \varphi_\xi(a_\xi) + b_\xi$. Writing (13) in terms of $\mathbb{T}_{\xi_k}$ we have

$$\mathbb{T}_{\xi_k}(l_i \cap \xi_k^\perp) = \Phi_{\xi_k}(l_i \cap \xi_k^\perp - l_i \cap \xi^\perp) + \Phi_{\xi_k}(l_i \cap \xi^\perp) + b_{\xi_k} = \tilde{l}_{\sigma(i)} \cap \xi_k^\perp.$$

Without loss of generality, both polytopes P and Q are located inside a large ball. Hence, the entries $\{a_{ij}^k\}_{k\in\mathbb{N}}$, $i, j = 1, 2, \dots, d$, of the matrix corresponding to the transformation Φ_{ξ_k}, and the coordinates b_j^k, $j = 1, 2, \dots, d$, of the vector b_{ξ_k} are bounded functions of ξ_k. By compactness, we can assume that $\{a_{ij}^k\}_{k\in\mathbb{N}}$ and b_j^k are convergent to the corresponding entries of $\tilde{\Phi}_\xi = \lim_{k\to\infty} \Phi_{\xi_k}$ and $\tilde{b}_\xi = \lim_{k\to\infty} b_{\xi_k}$ respectively. This yields

$$\tilde{\mathbb{T}}_\xi(l_i \cap \xi^\perp) = \tilde{\Phi}_\xi(l_i \cap \xi^\perp) + \tilde{b}_\xi = l_{\sigma(i)} \cap \xi^\perp,$$

where $\tilde{\mathbb{T}}_\xi = \lim_{k\to\infty} \mathbb{T}_{\xi_k}$. Hence, there exists $\tilde{T}_\xi$, such that the corresponding $\tilde{f}_\xi$ satisfies $\tilde{f}_\xi(l_i \cap \xi^\perp) = l_{\sigma(i)} \cap \xi^\perp$, $\forall i = 1, \dots, k$. This means that $\xi \in V_\sigma$ and V_σ is a closed set.

By the Baire category Theorem (we argue as in *Step 2* in the proof of Theorem 1), there exists a permutation σ_o such that the interior $\text{int}(V_{\sigma_o})$ is non-empty. Hence, (12) holds with $\sigma = \sigma_o$ and $U_o = \text{int}(V_{\sigma_o})$.

Step 3. By Lemma 4.1 the set of edges intersected by $\xi^\perp$, $\xi \in U_o$, coincides with the set of edges intersected by $\xi^\perp$, $\xi \in U_1$. To show that the corresponding lines in L and $\tilde{L}$ coincide up to a reflection in the origin, we will consider two cases: at least two lines in L are not parallel, or all lines in L are parallel.

Let $\xi \in U_o$ and let σ be as in (12). If at least two lines from L, say l_1, l_2, are not parallel, we apply the second part of Lemma 2.3 to the pair of lines (l_1, l_2) with vertices $v_1(\xi) = \xi^\perp \cap l_1$, $v_2(\xi) = \xi^\perp \cap l_2$ and the corresponding pair of lines $(\tilde{l}_{\sigma(1)}, \tilde{l}_{\sigma(2)}) \in \tilde{L}$ with vertices $\tilde{v}_{\sigma(1)}(\xi) = \xi^\perp \cap \tilde{l}_{\sigma(1)}$, $\tilde{v}_{\sigma(2)}(\xi) = \xi^\perp \cap \tilde{l}_{\sigma(2)}$. We see that the lines coincide up to a reflection in the origin.

Assume that $l_1 = \tilde{l}_{\sigma(1)}$ and $l_2 = \tilde{l}_{\sigma(2)}$. Then we choose any other line l_m in L, such that $l_m \nparallel l_1$ (if such l_m does not exists, choose $l_m \nparallel l_2$). Then we apply the second part of Lemma 2.3 to the pair (l_m, l_1). Recall that the permutation σ is the same in all $\xi^\perp$, such that $\xi \in U_1$, and we already have that $l_{\sigma(1)} = l_1$. This implies that $\tilde{l}_{\sigma(m)} = l_m$, since none of the lines passes through the origin, i.e., $l_i \neq -l_i$ or $\tilde{l}_j \neq -\tilde{l}_j$ for any i or j. We can repeat this argument in the case when $\tilde{l}_{\sigma(1)} = -l_1$ and $\tilde{l}_{\sigma(2)} = -l_2$ to conclude that $\tilde{l}_{\sigma(m)} = -l_m$ for any other line l_m from L. Thus,

$$l_j = \tilde{l}_{\sigma(j)} \quad \forall j = 1, \dots, k \quad \text{or} \quad l_j = -\tilde{l}_{\sigma(j)} \quad \forall j = 1 \dots, k. \tag{14}$$

Suppose now that for all $\xi \in U_1$, $\xi^\perp$ intersects only parallel lines in L. We claim that (14) holds in this case as well.

Consider a triple of lines $l_p, l_q, l_r \in L$, such that $v_i(\xi) = l_i \cap \xi^\perp, (i = p, q, r)$ are vertices of $P \cap \xi^\perp$. Notice that in this case the triple doesn't belong to a single two-dimensional plane. Apply Lemma 2.3 to the pairs l_p, l_q and l_p, l_r, on the open set $\xi \in U_1 \setminus (U_1 \cap Y_{pqr})$. Here Y_{pqr} is a nowhere dense subset obtained from applying Lemma 2.4 to the above triple. Assume that this yields

$$l_{\sigma(p)} = -l_p + c_{pq}, \quad l_{\sigma(q)} = -l_q + c_{pq}, \quad c_{pq} \in \mathbb{E}^d,$$

but

$$l_{\sigma(p)} = l_p + b_{pr}, \quad l_{\sigma(r)} = l_r + b_{pr}, \quad b_{pr} \in \mathbb{E}^d.$$

Then consider two triangles $v_p(\xi)v_q(\xi)v_r(\xi)$ and $\tilde{v}_{\sigma(p)}(\xi)\tilde{v}_{\sigma(q)}(\xi)\tilde{v}_{\sigma(r)}(\xi)$ in $\xi^\perp$ (see Figure 5). Notice that, by Lemma 2.4, angle $\angle v_q(\xi)v_p(\xi)v_r(\xi) \neq \frac{\pi}{2}$.

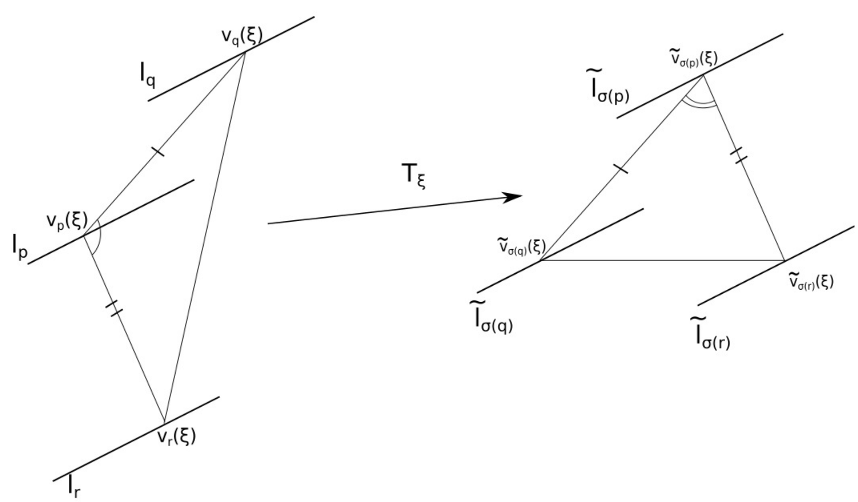

FIGURE 5. $T_\xi \left(v_p(\xi)v_q(\xi)v_r(\xi)\right) \neq \tilde{v}_{\sigma(p)}(\xi)\tilde{v}_{\sigma(q)}(\xi)\tilde{v}_{\sigma(r)}(\xi)$

On the other hand, since $\angle v_q(\xi)v_p(\xi)v_r(\xi) \neq \angle \tilde{v}_{\sigma(q)}(\xi)\tilde{v}_{\sigma(p)}(\xi)\tilde{v}_{\sigma(r)}(\xi)$, these triangles are not congruent under the fixed permutation σ, which contradicts the condition of congruency of $P \cap \xi^{\perp}$ and $Q \cap \xi^{\perp}$. Hence, for the triples we have

$$l_{\sigma(p)} = l_p + b_{pr}, \quad l_{\sigma(q)} = l_q + b_{pr}, \quad l_{\sigma(r)} = l_r + b_{pr}, \quad b_{pr} = b \in \mathbb{E}^d,$$

or

$$l_{\sigma(p)} = -l_p + c_{pr}, \quad l_{\sigma(q)} = -l_q + c_{pr}, \quad l_{\sigma(r)} = -l_r + c_{pr}, \quad c_{pr} = b \in \mathbb{E}^d.$$

Now we can repeat the same argument for any similar triples of lines in L and $\tilde{L}$. Since L contains a finite number of lines, we exclude a finite number of sets $\{Y_{pqr}\}_{l_p,l_q,l_r \in L}$ obtained from Lemma 2.4. Recall that each of these sets is closed and nowhere dense on S^{d-1}. We conclude that $\tilde{L} = L + b$ or $\tilde{L} = -L + b$. Assume that $\tilde{L} = L + b$, since we can always consider polytope $-Q$ instead of Q. We claim that $b = 0$.

Consider all edges of P that have a common vertex q with the edge $w_i \subset l_i \in L$, say, w_p, $p = 1, \dots, s(i)$.

Consider also all lines l_p containing q, and let $U_{i,p}$ be the corresponding non-empty connected component of U for which $span\{l_i, l_p\} \not\subset \xi^{\perp}$ for all $\xi \in U_{i,p}$ (since the interior of the solid angle A_q with vertex at q and edges $w_1, \dots, w_{s(i)}$ has dimension d, and $\dim \xi^{\perp} = d - 1$, this connected component always exists). In particular, $\dim(\mathrm{span}(l_i, l_p, O)) = 3$.

We apply the third part of Lemma 2.3 to the pairs (l_i, l_p) and $(\tilde{l}_{\sigma(i)}, \tilde{l}_{\sigma(p)})$ and the corresponding vertices $v_i(\xi) = \xi^{\perp} \cap l_i$, $v_p(\xi) = \xi^{\perp} \cap l_p$ and $\tilde{v}_{\sigma(i)}(\xi) = \xi^{\perp} \cap \tilde{l}_{\sigma(i)}$, $\tilde{v}_{\sigma(p)}(\xi) = \xi^{\perp} \cap \tilde{l}_{\sigma(p)}$. Here the lines l_i and $\tilde{l}_{\sigma(i)}$ belong to the corresponding sets of lines $L(U_{i,p})$ and $\tilde{L}(U_{i,p})$, and σ is the fixed permutation analogous to the one in (12).

If there exists i, such that for the line $l_i \in L$ Lemma 2.3 gives $l_i = \tilde{l}_{\sigma(i)}$, $l_p = \tilde{l}_{\sigma(p)}$, for all $p = 1, \dots, s(i)$, then $b = 0$. Indeed, in this case both polytopes P and Q have a common solid angle A_q containing q. Let $H \supset l_i$, $H \cap P = w_i$, be a supporting hyperplane to both P and Q with an inner (with respect to P) unit normal n_H. We will consider two cases, $n_H \cdot b > 0$ and $n_H \cdot b < 0$, and show that both are impossible (by changing H slightly we can assume that $n_H \cdot b \neq 0$). By the above, $P \subset \mathrm{conv}(l_1, \dots, l_k)$ and $Q \subset \mathrm{conv}(l_{\sigma(1)}, \dots, l_{\sigma(k)}) = b + \mathrm{conv}(l_1, \dots, l_k)$. If $n_H \cdot b > 0$, then Q should lie in the translated half-space $\mathcal{H} + b$, where $\mathcal{H}$ is the half-space with boundary H containing P and Q. Since Q contains w_i this is impossible, unless $b = 0$. If $n_H \cdot b < 0$, then Q contains A_q and $w_i + b$, which contradicts the convexity of Q. Hence, $b = 0$.

If for every $i = 1, \dots, k$ (for every line $l_i \in L$) there exists $p = p(i)$, $p = 1, \dots, s(i)$, for which Lemma 2.3 yields $l_i = -\tilde{l}_{\sigma(i)}$, $l_p = -\tilde{l}_{\sigma(p)}$, we have $l_i = -\tilde{l}_{\sigma(i)}$ for all $i = 1, \dots, k$.

Thus, (14) holds in the case when all lines in L are parallel as well.

Step 4. We proved that for any $\xi \in U_i$ the corresponding lines L and $\tilde{L}$ that are intersected by $\xi^\perp$ satisfy (14). This implies that for any vertex $v_i(\xi) \in P \cap \xi^\perp$ there exists a vertex $v_{\sigma(i)_\xi}(\xi) \in Q \cap \xi^\perp$ such that

$$v_i(\xi) = \tilde{v}_{\sigma(i)}(\xi) \quad \forall i = 1, \dots, k \quad \text{or} \quad v_i(\xi) = -\tilde{v}_{\sigma(i)}(\xi) \quad \forall i = 1, \dots, k.$$

Since any convex polytope is a convex hull of its vertices, we conclude that

$$Q \cap \xi^\perp = P \cap \xi^\perp \quad \forall \xi \in U \quad \text{or} \quad Q \cap \xi^\perp = -P \cap \xi^\perp \quad \forall \xi \in U.$$

This implies that for all $\xi \in U \setminus (Y \cup Y_1)$ the radial functions of P and Q satisfy $\rho_P(u) = \rho_Q(u)$ for all $u \in \xi^\perp$ or $\rho_P(-u) = \rho_Q(u)$ for all $u \in \xi^\perp$. Since the radial functions are continuous, we can prove that the same holds for any $\xi \in Y \cup Y_1$, and hence for all $\xi \in S^{d-1}$. Finally, we can apply Theorem 4 with $j = d-1$, $a_\alpha = 0$, to conclude that $\rho_P(u) = \rho_Q(u)$ for all $u \in S^{d-1}$ or $\rho_P(u) = \rho_Q(-u)$ for all $u \in S^{d-1}$(or see Lemma 1 in [R2]) . Theorem 2 is proved in the case $k = d-1$.

4.3. **Proof of Theorem 2 in the case $2 \le k < d-1$.** We use induction on k. Let H be any $(k+1)$-dimensional subspace of $\mathbb{E}^d$. We apply the result for $d = k+1$ to the bodies $P \cap H$ and $Q \cap H$ and their sections $(P \cap H) \cap J$ and $(Q \cap H) \cap J$ for all k-dimensional subspaces $J \subset H$. We obtain that $P \cap H = Q \cap H$ or $P \cap H = -Q \cap H$ for all H. We proceed and the result follows after finitely many steps. The proof of Theorem 2 is finished.

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