ON BODIES IN $\mathbb{R}^5$ WITH DIRECTLY CONGRUENT PROJECTIONS OR SECTIONS

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Abstract. Let K and L be two convex bodies in $\mathbb{R}^5$ with countably many diameters, such that their projections onto all 4 dimensional subspaces containing one fixed diameter are directly congruent. We show that if these projections have no rotational symmetries, and the projections of K, L on certain 3 dimensional subspaces have no symmetries, then $K = \pm L$ up to a translation. We also prove the corresponding result for sections of star bodies.

1. Introduction

In this paper we address the following problems (see [Ga, Problem 3.2, page 125 and Problem 7.3, page 289]).

Problem 1. *Suppose that $2 \le k \le n-1$ and that K and L are convex bodies in $\mathbb{R}^n$ such that the projection $K|H$ is congruent to $L|H$ for all $H \in \mathcal{G}(n,k)$. Is K a translate of $\pm L$?*

Problem 2. *Suppose that $2 \le k \le n-1$ and that K and L are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ is congruent to $L \cap H$ for all $H \in \mathcal{G}(n,k)$. Is K a translate of $\pm L$?*

Here we say that $K|H$, the projection of K onto H, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k,H)$ in H such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(n,k)$ stands for the Grassmann manifold of all k dimensional subspaces in $\mathbb{R}^n$.

Several partial results are known for Problems 1 and 2. For symmetric bodies, the answer is affirmative due to theorems of Aleksandrov (for Problem 1, see [A] and [Ga, Theorem 3.3.1, page 111]) and Funk (for Problem 2, see [Ga, Theorem 7.2.6, page 281]). In the class of polytopes, the answer to both problems is also affirmative [MyR]. If the projections are translates of each other, or if the bodies are convex and the corresponding sections are translates of each other, again a positive result is obtained (see [Ga, Theorems 3.1.3 and 7.1.1] and [R1]). For history and additional partial results, we refer the reader to [ACR], [My], [R], [R2].

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Hadwiger established that, for $n \geq 4$ and $k = n-1$, if the orthogonal transformations between the projections are all translations, it is not necessary to consider the projections onto all subspaces, but only all subspaces containing a fixed line (see [Ha], and [Ga, pages 126–127]).

In this paper, we obtain several Hadwiger-type results for both Problems when $k=4$ in the case of direct congruence; the fixed line will be given by the direction of one of the diameters of the body K. We follow the ideas from [Go], [R] and [ACR], where similar results were obtained in the cases $k=2,3$. The case $k=4$ is harder, due to the fact that four dimensional rotations are more difficult to handle than two or three dimensional ones. Nevertheless, here we obtain the expected conclusion of Problems 1 and 2 that $K = \pm L$ up to a translation, while in [ACR] the conclusion was that $K = L$ or $K = \mathcal{O}L$ up to a translation, for a certain orthogonal transformation $\mathcal{O}$ of $\mathbb{R}^n$.

We observe that the assumption about countability of the sets of the diameters of K and L can be weakened (for example, the set of diameters may be taken to be contained in a countable union of great circles containing ζ). Also, the set of bodies with countably many diameters contains the set of all polytopes whose four dimensional projections have no rigid motion symmetries, which is everywhere dense set in the class of convex bodies with respect to the Hausdorff metric ([Pa], see also [ACR, Proposition 2]).

1.1. **Results about directly congruent projections.** Let $n \geq 4$ and S^{n-1} be the unit sphere in $\mathbb{R}^n$. Given $w \in S^{n-1}$, let $w^\perp$ be the $(n-1)$ dimensional subspace of $\mathbb{R}^n$ that is orthogonal to w. We denote by $d_K(\zeta)$ a diameter of the body K which is parallel to the direction $\zeta \in S^{n-1}$.

Let D and B are two subsets of $H \in \mathcal{G}(n,k)$, $3 \leq k \leq n-1$. We say that D and B are directly congruent if $\varphi(D) = B + a$ for some vector $a \in H$ and some rotation $\varphi \in SO(k,H)$. We also say that D has an $SO(k)$ symmetry (respectively, $O(k)$ symmetry) if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-identical rotation $\varphi \in SO(k,H)$ (respectively, in $O(k,H)$).

We prove the following 5 dimensional result.

Theorem 1. *Let K and L be two convex bodies in $\mathbb{R}^5$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, such that the side projections $K|w^\perp$, $L|w^\perp$ onto all subspaces $w^\perp$ containing ζ are directly congruent, see Figure 1. Assume also that these projections have no $SO(4)$ symmetries, and that the three dimensional projections $K|(w^\perp \cap \zeta^\perp)$, $L|(w^\perp \cap \zeta^\perp)$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^5$.*

We state a generalization of Theorem 1 to n dimensions as a Corollary.

Corollary 1. *Let K and L be two convex bodies in $\mathbb{R}^n$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, such that the projections $K|H$, $L|H$ onto all four dimensional subspaces H containing ζ*

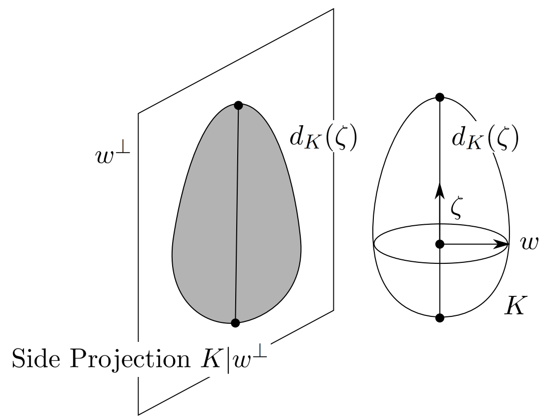

FIGURE 1. Diameter $d_K(\zeta)$ and side projection $K|w^\perp$.

are directly congruent. Assume also that these projections have no $SO(4)$ symmetries, and that the three dimensional projections $K|(H \cap \zeta^\perp)$ and $L|(H \cap \zeta^\perp)$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^n$.

1.2. **Results about directly congruent sections.** We also obtain results related to Problem 2.

Theorem 2. *Let K and L be two star bodies in $\mathbb{R}^5$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, containing the origin, such that the side sections $K \cap w^\perp$, $L \cap w^\perp$ by all subspaces $w^\perp$ containing ζ are directly congruent. Assume also that these sections have no $SO(4)$ symmetries, and that the three dimensional sections $K \cap (w^\perp \cap \zeta^\perp)$, $L \cap (w^\perp \cap \zeta^\perp)$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^5$ parallel to ζ.*

The n dimensional generalization of Theorem 2 is stated as a Corollary.

Corollary 2. *Let K and L be two star bodies in $\mathbb{R}^n$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, containing the origin, such that the sections $K \cap H$, $L \cap H$ by all four dimensional subspaces H containing ζ are directly congruent. Assume also that these sections have no $SO(4)$ symmetries, and that the three dimensional sections $K \cap H \cap \zeta^\perp$, $L \cap H \cap \zeta^\perp$ have no $O(3)$ symmetries. Then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{R}^n$ parallel to ζ.*

The paper is organized as follows. In Section 2, we introduce the needed definitions and notation. In Section 3, we prove the main auxiliary result of the paper, a functional equation similar to Proposition 1 in [ACR]. In Section 4 we prove Theorem 1 and Corollary 1, and in Section 5 we prove Theorem 2 and Corollary 2.

2. Notation and auxiliary definitions

We will use the following standard notation. The unit sphere in $\mathbb{R}^n$, $n \geq 2$, is S^{n-1}. Given $w \in S^{n-1}$, the hyperplane orthogonal to w and passing through the origin will be denoted by $w^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0\}$, where $x \cdot w = x_1 w_1 + \cdots + x_n w_n$ is the usual inner product in $\mathbb{R}^n$. The Grassmann manifold of all k dimensional subspaces in $\mathbb{R}^n$ will be denoted by $\mathcal{G}(n,k)$. The notation $O(k)$ and $SO(k)$, $2 \leq k \leq n$, for the subgroups of the orthogonal group $O(n)$ and the special orthogonal group $SO(n)$ in $\mathbb{R}^n$ is standard. If $\mathcal{U} \in O(n)$ is an orthogonal matrix, we will write $\mathcal{U}^t$ for its transpose.

We refer to [Ga, Chapter 1] for the next definitions involving convex and star bodies. A *body* in $\mathbb{R}^n$ is a compact set which is equal to the closure of its non-empty interior. A *convex body* is a body K such that for every pair of points in K, the segment joining them is contained in K. For $x \in \mathbb{R}^n$, the *support function* of a convex body K is defined as $h_K(x) = \max\{x \cdot y : \ y \in K\}$ (see page 16 in [Ga]). The *width function* $\omega_K(x)$ of K in the direction $x \in S^{n-1}$ is defined as $\omega_K(x) = h_K(x) + h_K(-x)$. A segment $[z,y] \subset K$ is called a *diameter* of the convex body K if $|z-y| = \max\limits_{\{\theta \in S^{n-1}\}} \omega_K(\theta)$. We say that a convex body $K \subset \mathbb{R}^n$ has countably many diameters if the width function ω_K reaches its maximum on a countable subset of S^{n-1}.

Observe that a convex body K has at most one diameter parallel to a given direction $\zeta \in S^{n-1}$ (for, if K had two parallel diameters d_1, d_2, then K would contain a parallelogram with sides d_1 and d_2, one of whose diagonals is longer than d_1). For this reason, if K has a diameter parallel to $\zeta \in S^{n-1}$, we will denote it by $d_K(\zeta)$.

A set $S \subset \mathbb{R}^n$ is said to be *star-shaped with respect to a point* p if the line segment from p to any point in S is contained in S. For $x \in \mathbb{R}^n \setminus \{0\}$, and $K \subset \mathbb{R}^n$ a nonempty, compact, star-shaped set with respect to the origin, the *radial function* of K is defined as $\rho_K(x) = \max\{c : \ cx \in K\}$. Here, the line through x and the origin is assumed to meet K ([Ga, page 18]). We say that a body K is a *star body* if it K is star-shaped with respect to the origin and its radial function ρ_K is continuous.

Given a star body K, a segment $[z,y] \subset K$ is called a *diameter* of K if $|z-y| = \max\limits_{\{[a,b] \subset K\}} |a-b|$. If a non-convex star body K has a diameter containing the origin that is parallel to $\zeta \in S^{n-1}$, we will also denote it by $d_K(\zeta)$.

Given $\zeta \in S^{n-1}$, the great $(n-2)$ dimensional sub-sphere of S^{n-1} that is perpendicular to ζ will be denoted by $S^{n-2}(\zeta) = \{\theta \in S^{n-1} : \theta \cdot \zeta = 0\}$. For $t \in [-1,1]$, the parallel to $S^{n-2}(\zeta)$ at height t will be denoted by $S^{n-2}_t(\zeta) = S^{n-1} \cap \{x \in \mathbb{R}^n : x \cdot \zeta = t\}$. Observe that when $t = 0$, $S^{n-2}_0(\zeta) = S^{n-2}(\zeta)$. Figure 2 shows the case $n = 5$.

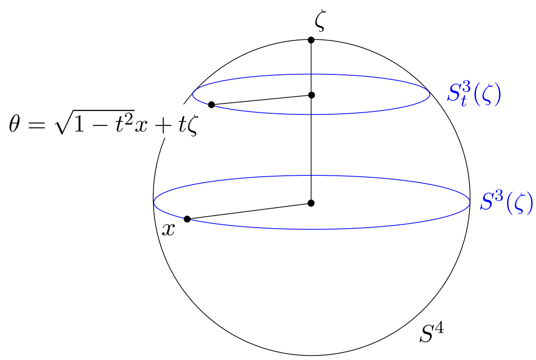

FIGURE 2. The great subsphere $S^3(\zeta)$ and the parallel $S^3_t(\zeta)$.

For $w \in S^4$, we will denote by $O(4, S^3(w))$, $SO(4, S^3(w))$, the orthogonal transformations in the 4 dimensional subspace spanned by the great subsphere $S^3(w)$ of S^4. The restriction of a transformation $\varphi \in O(n)$ onto the subspace of smallest dimension containing $W \subset S^{n-1}$ will be denoted by $\varphi|_W$. I stands for the identity transformation.

Finally, we define the notion of symmetry for functions, as it will be used throughout the paper.

Definition 1. *Let f be a continuous function on S^{n-1} and let $\xi \in S^{n-1}$. We say that the restriction of f onto $S^{k-1}(\xi)$ (or just f) has an $SO(k)$ symmetry if for some non-identical rotation $\varphi_\xi \in SO(k, S^{k-1}(\xi))$, we have $f \circ \varphi_\xi = f$ on $S^3(\xi)$. We similarly define the property that f has an $O(k)$ symmetry.*

3. A RESULT ABOUT A FUNCTIONAL EQUATION ON S^4

Proposition 1. *Let f and g be two continuous functions on S^4. Assume that for some $\zeta \in S^4$ and for every $w \in S^3(\zeta)$ there exists a rotation $\varphi_w \in SO(4, S^3(w))$, verifying that*

$$f \circ \varphi_w(\theta) = g(\theta), \quad \forall \theta \in S^3(w). \tag{1}$$

Assume, in addition, that $\varphi_w(\zeta) = \pm\zeta \ \ \forall w \in S^3(\zeta)$, that the restrictions of f and g to each $S^3(w)$ have no $SO(4)$ symmetries, and that the restrictions of f and g to each $S^3(w) \cap S^3(\zeta)$ have no $O(3)$ symmetries.

Then either $f = g$ on S^4 or $f(\theta) = g(-\theta)$ $\forall \theta \in S^4$.

3.1. **Auxiliary Lemmata.** We will divide the proof of Proposition 1 in several lemmata. The first Lemma describes the structure of the rotations φ_w that satisfy the the condition $\varphi_w(\zeta) = \pm\zeta$.

Lemma 1. *Let $\varphi \in SO(4)$ and let $\zeta \in \mathbb{R}^4$, $\zeta \neq 0$. If $\varphi(\zeta) = \pm\zeta$, then either $\varphi = \pm I$, or ζ belongs to one of the invariant 2 dimensional subspaces of φ, and the restriction of φ onto that subspace is a trivial rotation or a rotation by π.*

Proof. If $\varphi = \pm I$, the result is clear. Let $\varphi \neq \pm I$, and let Π and $\Pi^{\perp}$ be the two 2 dimensional invariant subspaces of φ. We can assume that the matrix of φ is written in the basis of pairwise orthogonal unit vectors $e_j \in \Pi$, $j = 1, 2$, and $e_j \in \Pi^{\perp}$, $j = 3, 4$. Then, for some $\alpha \in [0, 2)$ and $\beta \in [0, 2)$, our condition

$$\begin{bmatrix} \cos(\alpha\pi) & -\sin(\alpha\pi) & 0 & 0 \\ \sin(\alpha\pi) & \cos(\alpha\pi) & 0 & 0 \\ 0 & 0 & \cos(\beta\pi) & -\sin(\beta\pi) \\ 0 & 0 & \sin(\beta\pi) & \cos(\beta\pi) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix} = \pm \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix}$$

yields

$$\begin{bmatrix} \cos(\alpha\pi) & -\sin(\alpha\pi) \\ \sin(\alpha\pi) & \cos(\alpha\pi) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \pm \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \in \Pi$$

and

$$\begin{bmatrix} \cos(\beta\pi) & -\sin(\beta\pi) \\ \sin(\beta\pi) & \cos(\beta\pi) \end{bmatrix} \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} = \pm \begin{bmatrix} \zeta_3 \\ \zeta_4 \end{bmatrix} \in \Pi^{\perp}.$$

If both vectors (ζ_1, ζ_2) and (ζ_3, ζ_4) are non-trivial, then $\alpha = 0, 1$ and $\beta = 0, 1$, from which the result follows. On the other hand, if (ζ_1, ζ_2) is trivial, then α is arbitrary, $\beta = 0, 1$ and $\zeta \in \Pi^{\perp}$. Similarly, if (ζ_3, ζ_4) is trivial, then β is arbitrary and $\alpha = 0, 1$ and $\zeta \in \Pi$. $\square$

The next Lemma is an observation about the geometry of the sphere.

Lemma 2. *Let ζ and x be in S^k, $k \geq 3$. Then,*

$$\bigcup_{\{w \in S^{k-1}(\zeta) \cap S^{k-1}(x)\}} S^{k-1}(w) = S^k.$$

Proof. Let y be any point on S^k. Then $S^{k-1}(\zeta) \cap S^{k-1}(x) \cap S^{k-1}(y)$ is nonempty, since $k \geq 3$. Taking any $w \in S^{k-1}(\zeta) \cap S^{k-1}(x) \cap S^{k-1}(y) \subset S^{k-1}(\zeta) \cap S^{k-1}(x)$, it follows that $y \in S^{k-1}(w)$. $\square$

Lemma 3. *(cf. Lemma 1,* [R]*). Let $\zeta \in S^k$, $k \geq 4$. If for every $w \in S^{k-1}(\zeta)$ we have either $f(\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w)$ or $f(-\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w)$, then either $f = g$ on S^k or $f(-\theta) = g(\theta)$ for all $\theta \in S^k$.*

Proof. Assume at first that there exists an $x \in S^k$ such that for all $w \in S^{k-1}(\zeta) \cap S^{k-1}(x)$ we have $f(\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w)$. Then, using the previous lemma, we obtain $f = g$ on S^k.

Assume now that there exists an $x \in S^k$ such that for all $w \in S^{k-1}(\zeta) \cap S^{k-1}(x)$ we have $f(-\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w)$. Then, using the previous lemma, we obtain $f(-\theta) = g(\theta)$ for all $\theta \in S^k$.

Finally, assume that for every $x \in S^k$ there exist two directions w_1 and w_2 in $S^{k-1}(\zeta) \cap S^{k-1}(x)$ such that $f(\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w_1)$ and $f(-\theta) = g(\theta)$ for all $\theta \in S^{k-1}(w_2)$. Then $f(-x) = f(x) = g(x)$, and since x was chosen arbitrarily, we obtain $f = g$ on S^k. $\square$

For the next result, let $\mathcal{O} \in SO(5)$ be the orthogonal transformation defined by $\mathcal{O}(\zeta) = \zeta$ and $\mathcal{O}|_{S^3(\zeta)} = -I$. Observe that $\mathcal{O}|_{S^3(w)}$ commutes with every rotation $\varphi_w \in SO(4, S^3(w))$, such that $\varphi_w(\zeta) = \pm\zeta$, where $w \in S^3(\zeta)$. A function f defined on S^4 can be decomposed in the form

$$f(\theta) = \frac{f(\theta) + f(\mathcal{O}\theta)}{2} + \frac{f(\theta) - f(\mathcal{O}\theta)}{2} = f_{\mathcal{O},e}(\theta) + f_{\mathcal{O},o}(\theta), \quad \theta \in S^4, \tag{2}$$

and we will call $f_{\mathcal{O},e}$, $f_{\mathcal{O},o}$, the even and odd parts of f with respect to $\mathcal{O}$. Since $\mathcal{O}^2 = I$, we have

$$f_{\mathcal{O},e}(\theta) = f_{\mathcal{O},e}(\mathcal{O}\theta), \qquad f_{\mathcal{O},o}(\theta) = -f_{\mathcal{O},o}(\mathcal{O}\theta).$$

Given $y \in S^4$, we have that $y \in S^3_t(\zeta)$ for some $t \in [-1, 1]$, *i.e.* we can write

$$y = \sqrt{1-t^2}x + t\zeta, \tag{3}$$

for some $t \in [-1, 1]$ and $x \in S^3(\zeta)$ (see Figure 2). For any function f on S^4, we define the function F_t on $S^3(\zeta)$,

$$F_t(x) = F_{t,\zeta}(x) = f(\sqrt{1-t^2}x + t\zeta), \qquad x \in S^3(\zeta), \tag{4}$$

which is the restriction of f to $S^3_t(\zeta)$. Observe that the even part of F_t, $(F_t)_e$ equals

$$(F_t)_e(x) = \frac{f(\sqrt{1-t^2}x + t\zeta) + f(-\sqrt{1-t^2}x + t\zeta)}{2} = \frac{f(y) + f(\mathcal{O}y)}{2},$$

where y is as in (3), *i.e.*,

$$(F_t)_e(x) = f_{\mathcal{O},e}(y), \qquad (F_t)_o(x) = f_{\mathcal{O},o}(y). \tag{5}$$

Note that $(F_t)_e(x) = (F_t)_e(-x)$ for every $x \in S^3(\zeta)$. We similarly define G_t from the function g.

Every two dimensional great circle of $S^3(\zeta)$ is of the form $E_w := S^3(w) \cap S^3(\zeta)$ for some $w \in S^3(\zeta)$. Since $\varphi_w(\zeta) = \pm\zeta$ and $\varphi_w(S^3(w)) = S^3(w)$, we have

$$\varphi_w(E_w) = \varphi_w(S^3(w) \cap S^3(\zeta)) = S^3(w) \cap S^3(\zeta) = E_w.$$

For $\varphi_w \in SO(4, S^3(w))$ as in Proposition 1, we let $\phi_{E_w} = \varphi_w|E_w$ be the restriction of φ_w to E_w. Thus, for every $t \in [-1, 1]$, we have

$$F_t \circ \phi_{E_w}(x) = G_t(x) \qquad \forall x \in E_w, \tag{6}$$

Lemma 4. *Assume that f, g satisfy equation* (1) *for all $w \in S^3(\zeta)$. Then $f_{\mathcal{O},e}(y) = g_{\mathcal{O},e}(y)$ for every $y \in S^4$.*

Proof. For $w \in S^3(\zeta)$, we consider the spherical Radon transform

$$Rf(w) = \int_{E_w} f(x)dx$$

(see [Ga, pg. 429]). Since Lebesgue measure is invariant under orthogonal transformations on E_w, by (6) we have

$$\int_{E_w} F_t(x)dx = \int_{E_w} F_t(\varphi_w(x))dx = \int_{E_w} G_t(x)dx,$$

for each $t \in [-1,1]$. Hence, $RF_t(w) = RG_t(w)$ for every $w \in S^3(\zeta)$ and $t \in [-1,1]$, and it follows from Theorem C.2.4 from [Ga, pg. 430] that the even parts of F_t and G_t coincide. By equations (4) and (5), this means that $f_{\mathcal{O},e}(y) = g_{\mathcal{O},e}(y)$ for every $y \in S^4$.

$\square$

Note: Because of Lemma 4, from now on we will assume that f, g are odd with respect to $\mathcal{O}$.

Given $w \in S^3(\zeta)$ and $\varphi_w \neq \pm I$ verifying the hypotheses of Proposition 1, we have by Lemma 1 that $w^\perp = \Pi_w \oplus \Pi_w^\perp$, where $\zeta \in \Pi_w^\perp$, $\varphi_w|\Pi_w^\perp = \pm I$, and $\varphi_w|\Pi_w$ is a 2 dimensional rotation. On $\mathbb{R}^5$ we consider a positively oriented orthonormal basis $\{u, v, \zeta, z, w\}$, so that $\{u, v\}$ is a basis of Π_w, and $\{\zeta, z\}$ is a basis of $\Pi_w^\perp$. When we consider φ_w we mean that it is the restriction to the 4 dimensional subspace spanned by $S^3(w)$ of a rotation $\Phi \in SO(5)$ with the following properties: $\Phi(w) = w$, $\Phi|\Pi_w^\perp = \pm I$, and $\Phi|\Pi_w$ is a 2 dimensional rotation. Given $t \in \Pi_w \cap S^4$, if the angle between the vectors t and $\varphi_w(t) \in \Pi_w \cap S^4$ is $\alpha\pi$, for $\alpha \in (0,2)$, $\alpha \neq 1$, and $\{t, \varphi_w(t), \zeta, z, w\}$ forms a positively oriented basis of $\mathbb{R}^5$, then we will denote $\varphi_w|\Pi_w$ by $\varphi_w^{\alpha\pi}$ when we want to specify the angle of rotation.

We define the sets

$$\Xi_+ = \left\{ w \in S^3(\zeta) : \ f(\theta) = g(\theta) \ \forall \theta \in S^3(w) \right\},$$
$$\Xi_- = \left\{ w \in S^3(\zeta) : \ f(\theta) = g(-\theta) \ \forall \theta \in S^3(w) \right\},$$
$$\Xi_0 = \left\{ w \in S^3(\zeta) : \ f|\Pi_w = g|\Pi_w, \text{ and } f(\theta) = g(-\theta) \ \forall \theta \in \Pi_w^\perp \right\},$$
$$\Xi_1 = \left\{ w \in S^3(\zeta) : \ f|\Pi_w^\perp = g|\Pi_w^\perp, \text{ and } f(\theta) = g(-\theta) \ \forall \theta \in \Pi_w \right\},$$

and, for $\alpha \in (0,1) \cup (1,2)$,

$$\Xi_\alpha = \Big\{ w \in S^3(\zeta) : \varphi_w|\Pi_w^\perp = \pm I, \varphi_w|\Pi_w = \varphi_w^{\alpha\pi} \quad \text{and } f \circ \varphi_w(\theta) = g(\theta), \quad \forall \theta \in S^3(w) \Big\}. \tag{7}$$

With this notation, the hypothesis of Proposition 1 is that

$$S^3(\zeta) = \Xi_+ \cup \Xi_- \cup \bigcup_{\alpha \in [0,2)} \Xi_\alpha,$$

and we want to show that, under the condition on the lack of symmetries, we have either $S^3(\zeta) = \Xi_+$ or $S^3(\zeta) = \Xi_-$. By Lemma 3, this will imply that either $f = g$ on S^4 or $f(\theta) = g(-\theta)$ for all $\theta \in S^4$.

Lemma 5. *The sets Ξ_+, Ξ_-, Ξ_α are closed.*

Proof. Since the empty set is closed, we can assume that the sets Ξ_+, Ξ_- and Ξ_α are not empty. First we prove that Ξ_+ is closed. Let $(w_l)_{l=1}^\infty$ be a sequence of elements of Ξ_+ converging to $w \in S^3(\zeta)$ as $l \to \infty$, and let θ be any point on $S^3(w)$. Consider a sequence $(\theta_l)_{l=1}^\infty$ of points $\theta_l \in S^3(w_l)$ converging to θ as $l \to \infty$. (To see why such a sequence exists, see [ACR, Lemma 3]). By definition of Ξ_+ we have the following,

$$f(\theta_l) = g(\theta_l), \;\; \theta_l \in S^3(w_l), \;\; l \in \mathbb{N}.$$

Since f and g are continuous, we may pass to the limit and obtain $f(\theta) = g(\theta)$. Thus $w \in \Xi_+$ since the choice of $\theta \in S^3(w)$ was arbitrary, and hence Ξ_+ is closed. A similar proof shows that Ξ_- is closed, replacing $g(\theta_l)$ with $g(-\theta_l)$ and $g(\theta)$ with $g(-\theta)$.

Now we prove that Ξ_α is closed, where $\alpha \in [0, 2)$. As above, let $(w_l)_{l=1}^\infty$ be a sequence of elements of Ξ_α converging to $w \in S^3(\zeta)$ as $l \to \infty$, and let θ be any point on $S^3(w)$. Consider a sequence $(\theta_l)_{l=1}^\infty$ of points $\theta_l \in S^3(w_l)$ converging to θ as $l \to \infty$. From the hypothesis of the proposition, for each w_l there exists the rotation φ_{w_l}. Let Φ_{w_l} be the rotation in $\mathbb{R}^5$ whose restriction to $w_l^\perp$ is φ_{w_l}. By compactness, the sequence $\{\Phi_{w_l}\} \subseteq SO(5)$ has a convergent subsequence.

Suppose that $(\Phi_{w_l})_{l=1}^\infty$ has two subsequences that converge to two different rotations in $SO(5)$, $(\Phi_{w_j^1}) \to \Phi_{w^1}$ and $(\Phi_{w_k^2}) \to \Phi_{w^2}$, where $\Phi_{w^1} \neq \Phi_{w^2}$. Since w_l converges to w, and $\Phi_{w_l}(w_l) = w_l$, we have that $\Phi_{w^1}(w) = w, \Phi_{w^2}(w) = w$. Let φ_{w^1} be the restriction of Φ_{w^1} to the subspace $w^\perp$, and similarly $\varphi_{w^2} = \Phi_{w^2}|w^\perp$. We know that $f \circ \varphi_{w_j^1}(\theta_j) = g(\theta_j)$ and by passing to the limit we obtain that $f \circ \varphi_{w^1}(\theta) = g(\theta)$. Similarly, we have $f \circ \varphi_{w^2}(\theta) = g(\theta)$. This implies that $f \circ \varphi_{w^1}(\theta) = f \circ \varphi_{w^2}(\theta)$. Since the choice of θ was arbitrary, this last equation holds for all $\theta \in S^3(w)$. Thus, $f \circ \varphi_{w^1} \circ \varphi_{w^2}^{-1}(\theta) = f(\theta)$ for all $\theta \in S^3(w)$, where $\varphi_{w^1} \circ \varphi_{w^2}^{-1} \neq I$ since $\varphi_{w^1} \neq \varphi_{w^2}$. Thus, f has a $SO(4)$ symmetry on $S^3(w)$, which is a contradiction. Therefore, all convergent subsequences of $\{\Phi_{w_j}\}$ must have the same limit, which we will denote by Φ_w. It follows that the restrictions φ_{w_l} converge to $\varphi_w = \Phi_w|w^\perp$.

For each φ_{w_l} there is a unique pair of invariant two dimensional planes, namely Π_{w_l} and $\Pi_{w_l}^\perp$, with $\varphi_{w_l}|\Pi_{w_l}^\perp = \pm I$ and $\varphi_{w_l}|\Pi_{w_l} = \varphi_{w_l}^{\alpha\pi}$, since $w_l \in \Xi_\alpha$. Also, φ_w has a unique pair of invariant subspaces Π_w and $\Pi_w^\perp$, and therefore $(\Pi_{w_l}) \to \Pi_w$ and $(\Pi_{w_l}^\perp) \to \Pi_w^\perp$. Thus, there is either a subsequence $\varphi_{w_j^1}|\Pi_{w_j^1}^\perp = I$ (which would imply $\varphi_w|\Pi_w^\perp = I$), or a subsequence $\varphi_{w_j^1}|\Pi_{w_j^1}^\perp = -I$ (which implies that $\varphi_w|\Pi_w^\perp = -I$). Furthermore, since $\varphi_{w_l}|\Pi_{w_l}$ are rotations by the same angle $\alpha\pi$, for the limit we obtain that $\varphi_w|\Pi_w$ is also a rotation by the same angle, and since

$$f \circ \varphi_{w_l}^{\alpha\pi}(\theta_l) = g(\theta_l) \quad \theta_l \in \Pi_{w_l}, \quad l \in \mathbb{N}, \tag{8}$$

we conclude by continuity that $f \circ \varphi_w(\theta) = g(\theta)$. This shows that Ξ_α is closed.

□

The next Lemma shows that rotations by an irrational multiple of π do not occur because of the lack of symmetries of f and g.

Lemma 6. *Under the hypotheses of Proposition 1, we have* $\Xi_\alpha = \emptyset$ *for* $\alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 2)$.

Proof. Let $\alpha \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 2)$, and take $w \in \Xi_\alpha$. Following the ideas of Schneider [Sch1], we claim at first that $f^2 = g^2$ on $S^3(w)$. Indeed, since f and g are odd with respect to $\mathcal{O}$, f^2 and g^2 are even with respect to $\mathcal{O}$, and (1) holds with f^2, g^2 instead of f, g. Thus, by Lemma 4, we obtain that $f^2 = g^2$ on $S^3(w)$.

Squaring (1), we have

$$f^2 \circ \varphi_w(\theta) = g^2(\theta) = f^2(\theta) \qquad \forall \theta \in S^3(w).$$

Iterating, for any $k \in \mathbb{Z}$,

$$f^2 \circ \varphi_w^k(\theta) = f^2 \circ \varphi_w^{k-1}(\theta) = \cdots = f^2(\theta) \qquad \forall \theta \in S^3(w). \tag{9}$$

Let $\{\zeta, z\}$ be an orthonormal basis for $\Pi_w^\perp$, and consider the three dimensional subspace generated by Π_w and z, and its unit sphere $S^3(w) \cap S^3(\zeta)$. If $\varphi_w|\Pi_w^\perp = I$, then $\varphi_w|S^3(w) \cap S^3(\zeta)$ is a rotation of angle $\alpha\pi$ around the vector z. For each $\theta \in S^3(w) \cap S^3(\zeta)$, equation (9) holds for any $k \in \mathbb{Z}$, and since the orbit of $(\varphi_w^k(\theta))_{k\in\mathbb{Z}}$ is dense, we conclude that f^2 and g^2 are constant on each parallel of $S^3(w) \cap S^3(\zeta)$ perpendicular to z. By continuity, f and g must also be constant on each parallel, and thus $f \circ \varphi_w(\theta) = f(\theta)$ for every $\theta \in S^3(w) \cap S^3(\zeta)$. But then f has an $SO(3)$ symmetry on $S^3(w) \cap S^3(\zeta)$, contradicting the hypothesis of Proposition 1.

On the other hand, if $\varphi_w(\zeta) = -\zeta$, then $\varphi_w^2|S^3(w) \cap S^3(\zeta)$ is a rotation around z by the angle $2\alpha\pi$, and similarly to the previous case, f must be constant on every parallel of $S^3(w) \cap S^3(\zeta)$, and thus has a rotational symmetry on $S^3(w) \cap S^3(\zeta)$. This is a contradiction, and thus we have proven that $\Xi_\alpha = \emptyset$ for irrational α. □

Proof of Proposition 1. By Lemma 6, the sphere $S^3(\zeta)$ is equal to a countable union of closed sets,

$$S^3(\zeta) = \Xi_+ \cup \Xi_- \cup \bigcup_{\alpha\in[0,2)\cap\mathbb{Q}} \Xi_\alpha. \tag{10}$$

We observe that the sets on the right hand side of (10) are disjoint. Indeed, assume that there exists $z \in \Xi_\alpha \cap \Xi_\beta$, where α and β are either $+, -$ or a rational number in $[0, 2)$, and $\alpha \neq \beta$. Then, there are two rotations $\varphi_{z,\alpha}, \varphi_{z,\beta} \in SO(4, S^3(z))$ such that $f \circ \varphi_{z,\alpha}(\theta) = g(\theta)$ and $f \circ \varphi_{z,\beta}(\theta) = g(\theta)$ for all $\theta \in S^3(z)$. But then $f \circ \varphi_{z,\beta}(\theta) \circ (\varphi_{z,\alpha})^{-1}(\theta) = f(\theta)$ for all $\theta \in S^3(z)$, where $\varphi_{z,\beta}(\theta) \circ (\varphi_{z,\alpha})^{-1}$ is not the identity since $\alpha \neq \beta$. Thus, if $\Xi_\alpha \cap \Xi_\beta \neq \emptyset$, f has a symmetry on $S^3(z)$, contradicting the hypothesis of Proposition 1.

Thus, in (10) we have written $S^3(\zeta)$ as a countable union of disjoint closed sets. By a well-known result of Sierpiński's [Si], $S^3(\zeta)$ must equal just one of the sets. We will now assume that $S^3(\zeta) = \Xi_\alpha$ for some $\alpha \in [0, 2) \cap \mathbb{Q}$, and derive a contradiction. This will leave us only with the possibilities $S^3(\zeta) = \Xi_+$ or $S^3(\zeta) = \Xi_-$, and Proposition 1 will be proven.

Let $S^3(\zeta) = \Xi_\alpha$. Choose $w \in S^3(\zeta)$ and $\xi \in S^3(w) \cap S^3(\zeta)$. By Lemma 1, the subspace $\xi^\perp$ is equal to $\Pi_\xi \oplus \Pi_\xi^\perp$, with $\zeta \in \Pi_\xi^\perp$. Let $z_\xi \in \Pi_\xi^\perp$ be such that $\{\zeta, z_\xi\}$ is an orthonormal basis of $\Pi_\xi^\perp$.

We will write the set $S^3(w) \cap S^3(\zeta)$ as the union of two closed sets, Θ_{good} and Θ_{bad}, where

$$\Theta_{good} = \{\xi \in S^3(w) \cap S^3(\zeta) :\, \Pi_\xi = \xi^\perp \cap w^\perp \cap \zeta^\perp\},$$

$$\Theta_{bad} = \{\xi \in S^3(w) \cap S^3(\zeta) :\, \dim(\Pi_\xi \cap w^\perp \cap \zeta^\perp) = \dim(\Pi_\xi^\perp \cap w^\perp \cap \zeta^\perp) = 1\}.$$

Observe that Θ_{good} and Θ_{bad} are closed (this can be shown by an argument similar to the one in the proof of Lemma 5). They are also *disjoint* sets whose union equals $S^3(w) \cap S^3(\zeta)$. It follows that either $S^3(w) \cap S^3(\zeta) = \Theta_{good}$ or $S^3(w) \cap S^3(\zeta) = \Theta_{bad}$.

Assume that $S^3(w) \cap S^3(\zeta) = \Theta_{bad}$. We claim that, in this case, the map

$$\xi \to \ell(\xi) = \Pi_\xi^\perp \cap w^\perp \cap \zeta^\perp$$

defines a non-vanishing continuous tangent line field on the two dimensional sphere $S^3(w) \cap S^3(\zeta)$. If this map were not continuous, then there would exist two subsequences $\{\xi_j^1\}$ and $\{\xi_j^2\}$, both with limit ξ_0, such that

$$\lim_{j\to\infty} \ell(\xi_j^1) \neq \lim_{j\to\infty} \ell(\xi_j^2).$$

Denote by z_0^1 a unit vector in the direction of the line $\lim_{j\to\infty} \ell(\xi_j^1)$ and by z_0^2 a unit vector in the direction of the line $\lim_{j\to\infty} \ell(\xi_j^2)$. We have $z_0^1 \neq \pm z_0^2$. Let $\varphi_{\xi_0}^i = \lim\limits_{j\to\infty} \varphi_{\xi_j^i}$, for $i = 1, 2$, and let $\Pi_0^i \oplus (\Pi_0^i)^\perp$ the corresponding decompositions of $\xi_0^\perp$. Since all the rotations $\varphi_{\xi_j^i}|\Pi_{\xi_j^i}$ are by the angle $\alpha\pi$, the limiting rotations $\varphi_{\xi_0}^1|\Pi_{\xi_0^1}$ and $\varphi_{\xi_0}^2|\Pi_{\xi_0^2}$ are by the angle $\alpha\pi$ as well (this can be shown by a reasoning similar to the one in the proof of Lemma 5). But given that $z_0^1 \neq \pm z_0^2$, the subspaces $(\Pi_{\xi_j^1})^\perp$ and $(\Pi_{\xi_j^2})^\perp$ must be different. This means that $\varphi_{\xi_0}^1$ and $\varphi_{\xi_0}^2$ are two different rotations on $\xi_0^\perp$. From equation (1), it follows that $f \circ \varphi_{\xi_0}^1 \circ (\varphi_{\xi_0}^2)^{-1} = f$ on $S^3(\xi_0)$, where $\varphi_{\xi_0}^1 \circ (\varphi_{\xi_0}^2)^{-1} \neq I$. Therefore, we conclude that f has a rotational symmetry on $\xi_0^\perp$. Thus, the line field spanned by y_ξ must be continuous on the two dimensional sphere $S^3(w) \cap S^3(\zeta)$. This is impossible by a well known result of Hopf (see [Mi]).

We now consider the case in which $S^3(w) \cap S^3(\zeta) = \Theta_{good}$, *i.e.* the two dimensional space Π_ξ is equal to $\xi^\perp \cap w^\perp \cap \zeta^\perp$, and the restriction of φ_ξ to this subspace, which we will denote by ψ_ξ, is a rotation by the angle $\alpha\pi$. We have that the restrictions of f and g to $S^3(w)\cap S^3(\zeta)$ satisfy

$$f\circ\psi_\xi(\theta)=g(\theta),\ \forall\theta\in\Pi_\xi, \tag{11}$$

for every $\xi\in S^3(w)\cap S^3(\zeta)$. But every one dimensional great circle on the two dimensional sphere $S^3(w)\cap S^3(\zeta)$ is of the form $S^3(\xi)\cap S^3(w)\cap S^3(\zeta)$ for some $\xi\in S^3(w)\cap S^3(\zeta)$. We are thus under the hypothesis of the continuous Rubik's cube [R], and therefore we can conclude that either $f=g$ on $S^3(w)\cap S^3(\zeta)$, or $f(\theta)=g(-\theta)$ for every $\theta\in S^3(w)\cap S^3(\zeta)$.

Therefore, we have

$$f\circ\varphi_w(\theta)=g(\theta) \text{ and } f(\theta)=g(\theta)\quad\forall\theta\in S^3(w)\cap S^3(\zeta),$$

or

$$f\circ\varphi_w(\theta)=g(\theta) \text{ and } f(\theta)=g(-\theta)\quad\forall\theta\in S^3(w)\cap S^3(\zeta).$$

This implies that either

$$f\circ\varphi_w(\theta)=f(\theta)\quad\forall\theta\in S^3(w)\cap S^3(\zeta)$$

or

$$f\circ\varphi_w(\theta)=f(-\theta)\quad\forall\theta\in S^3(w)\cap S^3(\zeta),$$

where the restriction of φ_w to $S^3(w)\cap S^3(\zeta)$ is not the identity, since we are assuming that $w\in S^3(\zeta)=\Xi_\alpha$. Thus, the restriction of f to the 3 dimensional subspace spanned by $S^3(w)\cap S^3(\zeta)$ has an $O(3)$ symmetry, contradicting the hypothesis of Proposition 1.

Since the case $S^3(\zeta)=\Xi_\alpha$ leads to a contradiction, we conclude that either $S^3(\zeta)=\Xi_+$, or $S^3(\zeta)=\Xi_-$. Proposition 1 is proven.

□

4. Proof of Theorems 1 and 2

As in [ACR], the key ingredient in the proof of Theorem 1 is the existence of a diameter $d_K(\zeta)$ such that the side projections of K and L are directly congruent. We will first show that this implies that L must also have a diameter in the ζ direction, which necessarily has the same length as $d_K(\zeta)$. We can thus translate the bodies K and L so that their diameters $d_K(\zeta)$ and $d_L(\zeta)$ coincide and are centered around the origin. Since the translated bodies, $\tilde K$ and $\tilde L$, have countably many diameters, almost all side projections contain only this particular diameter, which must be fixed by the rotation. Therefore, we have reduced Theorem 1 to Proposition 1 with $f=h_{\tilde K}$ and $g=h_{\tilde L}$.

4.1. **Theorem 1 and Corollary 1.** Let $\zeta\in S^4$ be the direction of the diameter $d_K(\zeta)$ given in Theorem 1. By hypothesis, for every $w\in S^3(\zeta)$, the projections $K|w^\perp$ and $L|w^\perp$ are directly congruent. Hence, for every $w\in S^3(\zeta)$ there exists $\chi_w\in SO(4,S^3(w))$ and $a_w\in w^\perp$ such that

$$\chi_w(K|w^\perp)=L|w^\perp+a_w. \tag{12}$$

Let $\mathcal{A}_K \subset S^4$ be the set of directions parallel to the diameters of K, and $\mathcal{A}_L \subset S^4$ be the set of directions parallel to the diameters of L. We define

$$\Omega = \{w \in S^3(\zeta) : \quad (\mathcal{A}_K \cup \mathcal{A}_L) \cap S^3(w) = \{\pm\zeta\}\}. \tag{13}$$

The following two Lemmata are proven by the same arguments used in [ACR]. Lemma 7 shows that for most of the directions $w \in S^3(\zeta)$ the projections $K|w^\perp$ and $L|w^\perp$ have exactly one diameter, $d_K(\zeta)$ and $d_L(\zeta)$, respectively. We can thus translate the bodies K and L by vectors a_K, $a_L \in \mathbb{R}^5$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that their diameters $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ coincide and are centered at the origin.

Lemma 7. *(cf.* [ACR, Lemma 13]*.) Let K and L be as in Theorem 1, and let $\zeta \in \mathcal{A}_K$. Then $\zeta \in \mathcal{A}_L$, and Ω is everywhere dense in $S^3(\zeta)$. Moreover, for every $w \in \Omega$ we have $\chi_w(\zeta) = \pm\zeta$ and $\omega_K(\zeta) = \omega_L(\zeta)$.*

Lemma 8. *(cf.* [ACR, Lemma 14]*.) Let χ_w be the rotation given by (12), and let $w \in \Omega$. Then the rotation $\varphi_w := (\chi_w)^t$ satisfies $\varphi_w(\zeta) = \pm\zeta$ and*

$$h_{\tilde{K}} \circ \varphi_w(\theta) = h_{\tilde{L}}(\theta) \qquad \forall \theta \in S^3(w). \tag{14}$$

Proof of Theorem 1. Consider the closed sets $\Xi = \{w \in S^3(\zeta) :$ (14) holds with $\varphi_w(\zeta) = \zeta\}$ and $\Psi = \{w \in S^3(\zeta) :$ (14) holds with $\varphi_w(\zeta) = -\zeta\}$. Since the set $\Omega \subset (\Xi \cup \Psi)$ is everywhere dense in $S^3(\zeta)$ by Lemma 7, we have that $\Xi \cup \Psi = S^3(\zeta)$. We have thus reduced matters to Proposition 1 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$. Therefore, either $h_{\tilde{K}} = h_{\tilde{L}}$ on S^4 or $h_{\tilde{K}}(\theta) = h_{\tilde{L}}(-\theta)$ for every $\theta \in S^4$. This means that either $K + a_K = L + a_L$ or $K + a_K = -L - a_L$. $\square$

Proof of Corollary 1. First, we translate K and L by vectors $a_K, a_L \in \mathbb{R}^n$, obtaining the bodies $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$, so that the diameters $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ are centered at the origin. Next, we observe that for any five dimensional subspace J of $\mathbb{R}^n$, containing ζ, the bodies $\tilde{K}|J$ and $\tilde{L}|J$ verify the hypotheses of Theorem 1. Therefore, $\tilde{K}|J = \pm\tilde{L}|J$.

Assume that there exist two five dimensional subspaces J_1 and J_2, such that $\tilde{K}|J_1 = \tilde{L}|J_1$ and $\tilde{K}|J_2 = -\tilde{L}|J_2$. If $J_1 \cap J_2$ has dimension four, then

$$\tilde{L}|(J_1 \cap J_2) = (\tilde{L}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_2)|(J_1 \cap J_2)$$
$$= (-\tilde{L}|J_2)|(J_1 \cap J_2) = -\tilde{L}|(J_1 \cap J_2). \tag{15}$$

Since $-I \in SO(4)$, equation (15) implies that the projection $\tilde{L}|(J_1 \cap J_2)$ has an $SO(4)$ symmetry, contradicting the assumptions of the corollary. The same argument shows that if $J_1 \cap J_2$ is three dimensional, the projection $\tilde{L}|(J_1 \cap J_2)$ has an $O(3)$ symmetry (since $-I \in O(3)$). Next, assume that $J_1 \cap J_2$ is two dimensional, and let $\{\zeta, v_1, v_2, v_3, v_4\}$ and $\{\zeta, v_1, v_2', v_3', v_4'\}$ be orthonormal bases for J_1 and J_2, respectively. Consider the subspace J_0 spanned by $\{\zeta, v_1, v_2, v_2'\}$. Then, both $J_1 \cap J_0$ and $J_2 \cap J_0$ have dimension three, and the above argument can be used again. The case where $J_1 \cap J_2$

is one dimensional can be dealt with in a similar way. We conclude that either $\tilde{K}|J = \tilde{L}|J$ for every five dimensional subspace J containing ζ, or $\tilde{K}|J = -\tilde{L}|J$ for all such J. By Theorem 3.1.1 from [Ga, page 99], it follows that $\tilde{K} = \tilde{L}$ or $\tilde{K} = -\tilde{L}$. Thus, $K = L + a_L - a_K$ or $K = -L - a_L - a_K$. □

5. Proofs of Theorem 2 and Corollary 2.

We are now considering star-shaped bodies with respect to the origin. Let $\zeta \in S^4$ be the direction given in Theorem 2. The hypotheses imply that for every $w \in S^3(\zeta)$ there exists $\chi_w \in SO(4, S^3(w))$ and $a_w \in w^\perp$ such that

$$\chi_w(K \cap w^\perp) = (L \cap w^\perp) + a_w. \tag{16}$$

Let $l(\zeta)$ denote the one dimensional subspace containing ζ. As in Section 3, we let $\mathcal{A}_K \subset S^4$ be the set of directions that are parallel to the diameters of K (similarly for L). Note that it is possible for star-shaped bodies to contain several parallel diameters. We consider the set Ω, defined as in (13), and the set Ω^r, defined by

$$\Omega^r = \{w \in \Omega : K \cap w^\perp \text{ and } L \cap w^\perp \text{ have only one diameter}\}. \tag{17}$$

Then it follows from the hypothesis of Theorem 2 that if $w \in \Omega^r$, then $d_K(\zeta)$ must be the unique diameter of $K \cap w^\perp$. We will use the notation $v_K(\zeta) = \rho_K(\zeta) + \rho_K(-\zeta)$ for the length of the diameter $d_K(\zeta)$. As in the previous Section, it can be shown that for most directions $w \in S^3(\zeta)$, the sections $K \cap w^\perp$, $L \cap w^\perp$ contain exactly one diameter parallel to ζ and passing through the origin.

Lemma 9. *(cf.* [ACR, Lemma 15]*.) Let K and L be as in Theorem 2. Then L has a diameter $d_L(\zeta)$ passing through the origin, and Ω^r is everywhere dense in $S^3(\zeta)$. Moreover, for every $w \in \Omega^r$ we have $\chi_w(\zeta) = \pm\zeta$ and $v_K(\zeta) = v_L(\zeta)$.*

We now wish to argue as in the proof of Theorem 1 and translate the body L so that its diameter $d_L(\zeta)$, given by Lemma 9 coincides with $d_K(\zeta)$. However, the translate of a star-shaped body with respect to the origin may no longer be a star-shaped body with respect to the origin. The next Lemma, which is similar to Lemma 16 in [ACR] shows that, under our hypotheses, the translated body is still star-shaped (this Lemma is not necessary if K and L are convex). We include the proof here, since the rotation χ_w is now in $SO(4, S^3(w))$, and the argument is slightly different.

Lemma 10. *There exists a vector $a \in \mathbb{R}^5$, parallel to ζ, such that the body $\tilde{L} = L + a$ is star-shaped with respect to the origin, and $d_K(\zeta) = d_{\tilde{L}}(\zeta)$.*

Proof. Consider the sets

$$R_1 = \{w \in \Omega^r : \ \chi_w(d_K(\zeta)) = d_K(\zeta)\},$$
$$R_2 = \{w \in \Omega^r : \ \chi_w(d_K(\zeta)) \neq d_K(\zeta)\},$$

where χ_w is the rotation in $SO(4, S^3(w))$ as in (16). By Lemma 1, since $\chi_w(\zeta) = \pm\zeta$, we have that either $\chi_w = \pm I$, or that χ_w has an invariant two dimensional subspace $\Pi^\perp$, containing ζ, such that $\chi_w|\Pi^\perp = \pm I$. If $\chi_w = I$ or $\chi_w|\Pi^\perp = I$, or if $d_K(\zeta)$ is centered at the origin, then $w \in R_1$. The only case in which $w \in R_2$ is if $d_K(\zeta)$ is not centered at the origin, and either $\chi_w = -I$ or $\chi_w|\Pi^\perp = -I$.

Assume, at first, that $\Omega^r = R_1$. Since the diameter $d_K(\zeta)$ is fixed by χ_w, and $d_L(\zeta)$ contains the origin, it follows that the vector a_w in (16) is independent of $w \in \Omega^r$ and $a_w = a_1 = (\rho_K(\zeta) - \rho_L(\zeta))\,\zeta$. The translated section $(L\cap w^\perp)+a_1$ coincides with $\chi_w(K\cap w^\perp)$, and therefore $(L\cap w^\perp)+a_1$ is star-shaped with respect to the origin for every $w \in \Omega^r$. Since Ω^r is dense in $S^3(\zeta)$, we conclude that the translated body $\tilde{L} = L + a$, with $a = a_1$, is also star-shaped with respect to the origin.

Secondly, assume that $\Omega^r = R_2$. Then, a_w is independent of $w \in \Omega^r$ and $a_w = a_2 = (\rho_K(-\zeta) - \rho_L(\zeta))\,\zeta$. We conclude that $\tilde{L} = L + a$, with $a = a_2$, is star-shaped with respect to the origin.

Finally, we show that the case where R_1 and R_2 are both nonempty does not occur under the assumptions of Theorem 2. Since $R_1 \cup R_2 = \Omega^r$, we have $S^3(\zeta) = \overline{R_1 \cup R_2} \subseteq \overline{R_1} \cup \overline{R_2} \subseteq S^3(\zeta)$. Hence, there exists $w_0 \in \overline{R_1} \cap \overline{R_2}$, *i.e.*, there is a rotation χ_{w_0} such that $\chi_{w_0}(d_K(\zeta)) = d_K(\zeta)$ and

$$\chi_{w_0}(K \cap w_0^\perp) = L \cap w_0^\perp + a_1, \tag{18}$$

and a rotation $\tilde{\chi}_{w_0}$ such that $\tilde{\chi}_{w_0}(d_K(\zeta)) \neq d_K(\zeta)$ and

$$\tilde{\chi}_{w_0}(K \cap w_0^\perp) = L \cap w_0^\perp + a_2. \tag{19}$$

In particular, since $\tilde{\chi}_{w_0}$ does not fix $d_K(\zeta)$, this diameter cannot be centered at the origin, and it follows that the other rotation χ_{w_0} must be the identity, at least on a two dimensional subspace containing ζ. By (18) and (19) we have

$$K \cap w_0^\perp = \chi_{w_0}^{-1}\left(\tilde{\chi}_{w_0}(K \cap w_0^\perp)\right) + b,$$

where $b \in \mathbb{R}^5$. Observe that the rotation $\chi_{w_0}^{-1} \circ \tilde{\chi}_{w_0}$ is not the identity, since $\chi_{w_0}^{-1} \circ \tilde{\chi}_{w_0}(\zeta) = -\zeta$. Therefore, $K \cap w_0^\perp$ has a rotational symmetry. This contradicts the hypothesis of Theorem 2. The Lemma is proven. □

In order to finish the argument, we need one further Lemma.

Lemma 11. *(cf.* [ACR, Lemma 17]*.) For every $w \in \Omega^r$ there exists $\varphi_w = \chi_w^{-1} \in SO(4, S^3(w))$, $\varphi_w(\zeta) = \pm\zeta$, such that*

$$\rho_K \circ \varphi_w(\theta) = \rho_{\tilde{L}}(\theta) \qquad \forall \theta \in S^3(w). \tag{20}$$

Proof of Theorem 2. Consider the sets

$$\Xi^r = \{w \in S^3(\zeta) : \quad (20) \quad \text{holds with} \quad \varphi_w(\zeta) = \zeta\}$$

and

$$\Psi^r = \{w \in S^3(\zeta) : \quad (20) \quad \text{holds with} \quad \varphi_w(\zeta) = -\zeta\}.$$

By definition, $\Omega^r \subset (\Xi^r \cup \Psi^r)$. Therefore, Lemma 9 implies that $\Xi^r \cup \Psi^r = S^3(\zeta)$. Now we can apply Proposition 1 (with $f = \rho_K$, $g = \rho_{\tilde{L}}$, and $\Xi = \Xi^r$, $\Psi = \Psi^r$) obtaining that either $\rho_K = \rho_{\tilde{L}}$ on S^4, or $\rho_K(\theta) = \rho_{\tilde{L}}(-\theta)$ for all $\theta \in S^4$. In the first case, $K = \tilde{L}$, and in the second, $K = -\tilde{L}$. Thus, either $K = L + a$, or $K = -L - a$. This finishes the proof of Theorem 2. □

Proof of Corollary 2

The proof is similar to the one of Corollary 1. One has only to consider the sections $K \cap J$, $\tilde{L} \cap J$, instead of the projections $K|J$, $\tilde{L}|J$, and Theorem 7.1.1 from [Ga, page 270], instead of Theorem 3.1.1 from [Ga, page 99]. □

References

[A] A. D. Alexandrov, *On the theory of mixed volumes of convex bodies II. New inequalities between mixed volumes and their applications* [In Russian], Mat. Sbornik, 2 (1937), 1205-1238.

[ACR] M. A. Alfonseca, M. Cordier, D. Ryabogin, *On bodies with directly congruent projections and sections*, Israel J. Math. 215 (2016), no. 2, 765–799.

[Ga] R. J. Gardner, *Geometric tomography, second edition.* Encyclopedia of Mathematics and its Applications, 58. Cambridge University Press, Cambridge, 2006.

[Go] V. P. Golubyatnikov, *Uniqueness questions in reconstruction of multidimensional objects from tomography type projection data. Inverse and Ill-posed problems series.* De Gruyter, Utrecht-Boston-Köln-Tokyo, 2000.

[Ha] H. Hadwiger, *Seitenrisse konvexer Körper und Homothetie*, Elem. Math. 18 (1963), 97-98.

[Mi] J. Milnor, *Analytic proofs of the "Hairy ball theorem" and the Brouwer fixed point theorem*, Amer. Math. Monthly 85 (1978), no. 7, 521-524.

[My] S. Myroshnychenko, *On a functional equation related to a pair of hedgehogs with congruent projections or sections*, J. Math. Anal. Appl. 445 (2017), no. 2, 1492–1504.

[MyR] S. Myroshnychenko and D. Ryabogin, *On polytopes with congruent projections and sections*, submitted.

[Pa] R. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) 73 (1961), 295-323.

[R] D. Ryabogin, *On the continual Rubik's cube*, Adv. Math. 231 (2012), 3429-3444.

[R1] D. Ryabogin, *A Lemma of Nakajima and Süss on convex bodies*, Amer. Math. Monthly 122 (2015), no. 9, 890-892.

[R2] D. Ryabogin, *On symmetries of projections and sections of convex bodies*, Springer Contributed Volume "Discrete Geometry and Symmetry" dedicated to Karoly Bezdek and Egon Schulte on the occasion of their 60-th birthdays, to appear.

[Sch1] R. Schneider, *Convex bodies with congruent sections*, Bull. London Math. Soc. 312 (1980), 52-54.

[Si] W. Sierpiński, *Un théorèm sur les continus*, Tohoku Math. J. 13 (1918), 300–303.

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