HARDY SPACES AND PARTIAL DERIVATIVES OF CONJUGATE HARMONIC FUNCTIONS

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Abstract. In this paper we give necessary and sufficient conditions for a harmonic vector and all its partial derivatives to belong to $H^p(\mathbf{R}^{n+1}_+)$ for all $p > 0$.

1. Introduction and statements of main results

In this article we study the following problem: what can we say about conjugate harmonic functions in $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times (0, \infty)$, provided we are given certain restrictions, imposed on partial derivatives of a harmonic vector F,

$$F = (U(x,y), V_1(x,y), V_2(x,y), ..., V_n(x,y)), \qquad (x,y) \in \mathbf{R}^{n+1}_+.$$

We refer the reader to the classical works [11], [3], [13], [15], [5], [2], [4], [16] for the history and different results related to this problem and classes $S^p(\mathbf{R}^{n+1}_+), h^p(\mathbf{R}^{n+1}_+)$, $H^p(\mathbf{R}^{n+1}_+)$, (all definitions are given in Section 2).

We give necessary and sufficient conditions for a harmonic vector and all its partial derivatives up to the order k to belong to $H^p(\mathbf{R}^{n+1}_+)$, $p > 0$. Our main result is

Theorem 1. *Let $0 < p < q$. The harmonic vector $F = (U, V_1, ..., V_n)$ and all its partial derivatives of the order $\leq k$ belong to H^r, $p \leq r \leq q$, if and only if*

$$(1) \qquad 1)\ M_p(y+1, F) \leq C, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \geq y} |D^k_{n+1} U(x,\eta)| \Big)^q dx \leq C.$$

The case $p < (n-1)/n$ leads to additional technical difficulties, since $|F|^p$ is subharmonic, provided $p \geq (n-1)/n$, [13]. One of the methods of the proof is the application of classes $S^p(\mathbf{R}^{n+1}_+)$ together with the Lagrange mean-value Theorem. We also use the boundary behaviour of the conjugate harmonic functions. We note that the first condition in (1) is natural not only due to the decomposition of the function into two parts, "$S^p(\mathbf{R}^{n+1}_+)$" and "$H^p(\mathbf{R}^{n+1}_+)$" (see Section 3, Proof of Theorem 5). In fact, it, together with the second condition, implies $M_p(y + y_0, F) \leq C(y_0)\ \forall y_0 > 0$ (see Section 4, Lemma 7).

The paper is organized as follows. In section 2 we give all necessary definitions and auxiliary results used in the sequel. Section 3 is devoted to the results needed for the proof of Theorem 1, and in section 4 we prove Theorem 1. For convenience of the reader we split our proofs into elementary Lemmata.

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2. Auxiliary results

We begin with the definition of classes $S^p(\mathbf{R}^{n+1}_+)$ and $h^p(\mathbf{R}^{n+1}_+)$.

Let $U(x,y)$ be a harmonic function in $\mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times (0,\infty)$. We say that the vector-function $V(x,y) = (V_1(x,y),...,V_n(x,y))$ is the conjugate of $U(x,y)$ in the sense of M. Riesz [12], [14], if $V_k(x,y)$, $k=1,...,n$ are harmonic functions, satisfying the generalized Cauchy-Riemann conditions:

$$\frac{\partial U}{\partial y} + \sum_{k=1}^{n} \frac{\partial V_k}{\partial x_k} = 0, \qquad \frac{\partial V_i}{\partial x_k} = \frac{\partial V_k}{\partial x_i}, \qquad \frac{\partial U}{\partial x_i} = \frac{\partial V_i}{\partial y}, \qquad i \neq k,\ k=1,...,n.$$

If $U(x,y)$ and $V(x,y)$ are conjugate in $\mathbf{R}^{n+1}_+$ in the above sense, then the vector-function

$$F(x,y) = (U(x,y), V(x,y)) = (U(x,y), V_1(x,y),...,V_n(x,y))$$

is called a harmonic vector.

Define

$$M_p(y) = M_p(y,F) = \Big\{ \int_{\mathbf{R}^n} |F(x,y)|^p dx \Big\}^{1/p}, \qquad p>0.$$

Definition 1 ([1], [7]). *We say that $F(x,y) \in S^p(\mathbf{R}^{n+1}_+)$, $p>0$ if for any $y_0>0$ there exists a constant $C(y_0,F)$, such that $\forall y \geq y_0$, $M_p(y,F) \leq C(y_0)$. In particular, if C is independent of y_0, then $F(x,y) \in h^p(\mathbf{R}^{n+1}_+)$.*

Now we define the space $H^p(\mathbf{R}^{n+1}_+)$. We follow the work of Fefferman and Stein[4]. Let $U(x,y)$ be a harmonic function in $\mathbf{R}^{n+1}_+$, and let $U_{j_1 j_2 j_3 \dots j_k}$ denote a component of a symmetric tensor of rank k, $0 \leq j_i \leq n$, $i=1,...,n$. Suppose also that the trace of our tensor is zero, meaning

$$\sum_{j=0}^{n} U_{jjj_3\dots j_k}(x,y) = 0, \qquad \forall j_3,...,j_k.$$

The tensor of rank $k+1$ can be obtained from the above tensor of rank k by passing to its gradient:

$$U_{j_1 j_2 \dots j_k j_{k+1}}(x,y) = \frac{\partial}{\partial x_{j_{k+1}}}(U_{j_1 j_2 j_3 \dots j_k}(x,y)), \qquad x_0 = y,\ 0 \leq j_{k+1} \leq n.$$

Definition 2 ([4]). *We say that $U(x,y) \in H^p(\mathbf{R}^{n+1}_+)$, $p>0$, if there exists a tensor of rank k of the above type with the properties:*

$$U_{0\dots0}(x,y) = U(x,y), \qquad \sup_{y>0} \int_{\mathbf{R}^n} \Big(\sum_{(j)} U^2_{(j)}(x,y) \Big)^{p/2} dx < \infty, \qquad (j) = (j_1,...j_k).$$

It is well-known that the function $\Big(\sum_{(j)} U^2_{(j)}(x,y)\Big)^{p/2}$ is subharmonic for $p \geq p_k = (n-k)/(n+k-1)$, see [3],[4],[14].

We will use the "radial" and nontangential maximal functions:

$$F^+(x) = \sup_{y>0} |F(x,y)|, \qquad N_\alpha(F)(x^0) = \sup_{(x,y)\in\Gamma_\alpha(x^0)} |F(x,y)|.$$

Here

$$\Gamma_\alpha(x^0) = \{(x,y) \in \mathbf{R}^{n+1}_+ : |x - x^0| < \alpha y\}, \qquad \alpha > 0,$$

is an infinite cone with the vertex at x^0. It is well-known [4] that

$$F(x,y) \in H^p(\mathbf{R}^{n+1}_+) \iff N_\alpha(F)(x) \in L^p \iff F^+(x) \in L^p, \ p > 0.$$

We also define the *weak maximal function*

$$WF(x,y) = \sup_{\zeta \geq y} |F(x,\zeta)|, \qquad y > 0.$$

The above expression is understood as folows: we fix x, and for fixed y we find the supremum over all $\zeta \geq y$.

We will repeatedly use the following results.

Lemma 1. *([4], p.173). Suppose $u(x,y)$ is harmonic in $\mathbf{R}^{n+1}_+$, and for some p, $0 < p < \infty$,*

$$\sup_{y>0} \int_{\mathbf{R}^n} |u(x,y)|^p dx < \infty,$$

then

$$\sup_{x \in \mathbf{R}^n} |u(x,y)| \leq A y^{-n/p}, \qquad 0 < y < \infty. \tag{2}$$

Theorem 2. *([5], p. 268). Let $0 < p \leq 1$, $k \in \mathbf{N}$, and let $u : \mathbf{R}^{n+1}_+ \to \mathbf{R}$ be a harmonic function such that*

$$u(x,t) \rightrightarrows^x_{t\to\infty} 0, \qquad K_{k,p} \equiv \int_{\mathbf{R}^{n+1}_+} t^{kp-1} |D^k_{n+1} u(x,t)|^p dx dt < C.$$

Then $u(x,0) = \lim_{t\to 0+} u(x,t)$ exists for almost all $x \in \mathbf{R}^n$, and for all $t \geq 0$,

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \leq AC(k,n,p) K_{k,p}.$$

Theorem 3. *([5], p. 269). Let $m \in \mathbf{N}$, $p \geq (n-1)/(m+n-1)$ (if $n = 1$ we suppose $p > 0$), and let $u : \mathbf{R}^{n+1}_+ \to \mathbf{R}$ be harmonic. Then, for all $t > 0$,*

$$\int_{\mathbf{R}^n} |\nabla^m u(x,t)|^p dx \leq A(m,n,p) t^{-mp-1} \int_{t/2}^{3t/2} ds \int_{\mathbf{R}^n} |u(x,t)|^p dx.$$

Corollary 1. *([5], p. 270). Let p, m be as in Theorem 3, let $b > 0$, and let $u : \mathbf{R}^{n+1}_+ \to \mathbf{R}$ be a harmonic function such that for all $t > 0$*

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \leq Ct^{-b}.$$

Then

$$\int_{\mathbf{R}^n} |\nabla^m u(x,t)|^p dx \leq A(b,m,n,p) C t^{-b-mp}, \qquad (t > 0).$$

In fact, the choice of p in Theorem 2 and Corollary 1 may be independent of m (see Lemma 2).

Theorem 4. [10]. *Let $p > 0$ and let $F(x,y) = (U, V_1, ..., V_n)$ be a harmonic vector satisfying*

$$(3) \qquad 1)\, V_i(x,y) \rightrightarrows^x_{y\to\infty} 0,\ i = 1,...,n, \qquad 2)\, M_p(y,U) \le C, \qquad 3)\, |U| \le C.$$

Then $F \in H^r$, $r > p$.

Notation. We denote by $D^k_i f(x,y)$ the partial derivative of the function f of the order k with respect to x_i, $i = 1, 2, ..., n+1$. $M(f)(x)$ denotes the usual Hardy-Littlewood maximal function of $f(x)$. The notation $f(x,y) \rightrightarrows^x_{y\to\infty} 0$ means that $f(x,y)$ converges to 0 uniformly with respect to x, provided $y \to \infty$, $\nabla^k f(x) = (\frac{\partial^k f(x)}{\partial x_1^k}, ..., \frac{\partial^k f(x)}{\partial x_n^k})$. Everywhere below the constants $A(k,n), C, K$ depend only on the parameters pointed in parentheses, and may be different from time to time.

3. Auxiliary lemmata for the proof of Theorem 1.

The main results of this section are Theorem 5 and Theorem 7. Our first auxiliary result shows that in Theorem 3 and Corollary 1 the choice of $p > 0$ may be independent on $m \in \mathbf{N}$. We include it here for convenience of the reader.

Lemma 2. *Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be such that $V_i \rightrightarrows^x_{y\to\infty} 0$, $i = 1, ..., n$, $M_p(y,U) \le C$. Then*

$$M_p(y, \nabla^k F) \le ACy^{-k}, \qquad k \in \mathbf{N}.$$

Proof. By induction on k. Let $k = 1$. Fix $p > 0$ and let $l = \inf\{j \in \mathbf{N} : p \ge (n-1)/(j+n-1)\}$. Let $\phi_{ij}(x,y)$ be a coordinate of $\nabla V_i(x,y)$, $j = 1, ..., n+1$, $x_{n+1} = y$, $i = 0, ..., n$, $V_0 = U$. Since $\nabla V_i(x,y) \rightrightarrows^x_{y\to\infty} 0$, we may use the following relation (see [5] or [4])

$$\phi_{ij}(x,y) = \frac{1}{(2l-2)!}\int_y^\infty (s-y)^{2l-2} D^{2l-1}_{n+1}\phi_{ij}(x,s)ds =$$

$$\frac{1}{(2l-2)!}\int_0^\infty s^{2l-2} D^{2l-1}_{n+1}\phi_{ij}(x,s+y)ds.$$

We have $|\phi_{ij}(x,y)| \le h_{ij}(x,y)$, where

$$h_{ij}(x,y) \equiv \frac{1}{(2l-2)!}\int_0^\infty s^{2l-2}|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+y)|^p ds.$$

Theorem 3 of[5] implies (take $w = \nabla^l D^{l-1}_{n+1}\phi_{ij}$, $a = 2l-1$, $A = A(l,n,p)$),

$$\int_{\mathbf{R}^n} |\phi_{ij}(x,y)|^p dx \le \int_{\mathbf{R}^n} |h_{ij}(x,y)|^p dx \le A\int_0^\infty s^{(2l-1)p-1}ds \int_{\mathbf{R}^n} |\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+y)|^p dx.$$

Since $D_{n+1}^{l-1}\phi_{ij}(x,y)$ is the $l-$th derivative of V_i, we use Theorem 3 to get

$$\int\limits_{\mathbf{R}^n} |\nabla^l D_{n+1}^{l-1}\phi_{ij}(x,y)|^p dx \le \int\limits_{\mathbf{R}^n} |\nabla^{2l} F(x,y)|^p dx \le C\, y^{-2lp}. \tag{4}$$

This gives

$$\int\limits_{\mathbf{R}^n} |\phi_{ij}(x,y)|^p dx \le A(l,n,p)C \int\limits_0^\infty s^{(2l-1)p-1}(s+y)^{-2lp} ds = A(l,n,p)Cy^{-p},$$

and the first induction step is proved.

Assume that the statement is true for $k-1$. Then $M_p(y,\nabla^{k-1}F) \le ACy^{-(k-1)}$. To prove it for k we define l as above and apply Corollary 1 with $b=k-1$, $m=1$, $u=\nabla^{k-1}F$. □

Lemma 3. *Let* $D_i U \rightrightarrows^x_{y\to\infty} 0$, $i=1,...,n$, *and let*

$$M_p(y, D_{n+1}U) \le Cy^{-1} \tag{5}$$

for some $p>0$. *Then*

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}U(x,y+y_0)|\Big)^p dx \le AC, \qquad \forall y_0>0,\ \ A=A(n,p,y_0). \tag{6}$$

Proof. Let $p>1$. Then (see [12])

$$\| \sup_{y>0} |D_{n+1}U(\cdot,y+y_0)|\|_p \le CM_p(y+y_0, D_{n+1}U) \le AC, \qquad A=A(n,p,y_0).$$

Now let $0<p\le 1$. Assume that

$$\int\limits_0^\infty \int\limits_{\mathbf{R}^n} s^{p-1}|\nabla^2 U(x,s+y_0)|^p dxds \equiv \int\limits_0^1\int\limits_{\mathbf{R}^n} + \int\limits_1^\infty\int\limits_{\mathbf{R}^n} < C(y_0) < \infty. \tag{7}$$

Then Lemma 2, Theorem 2 (with $D_{n+1}U$ instead of u and $k=1$), and the tensor representation of $D_{n+1}U$ from [4], imply $D_{n+1}U(x,y+y_0)\in H^p$. This gives (6).

Thus, we have to show (7). By Theorem 3, (5) yields

$$M_p(y,\nabla^2 U) \le ACy^{-2}. \tag{8}$$

Then, the first integral in the right-hand side of (7) is finite, since

$$M_p(y+y_0,\nabla^2 U) \le AC(y+y_0)^{-2} \le AC(y_0), \qquad \forall y_0>0.$$

On the other hand,

$$\int\limits_1^\infty s^{p-1} ds \int\limits_{\mathbf{R}^n} |\nabla^2 U(x,s+y_0)|^p dx \le AC \int\limits_1^\infty s^{p-1}(s+y_0)^{-2p} ds \le AC(y_0) < \infty.$$

□

Let V_i be components of the harmonic vector $F = (U, V_1, ..., V_n)$. By the mean-value theorem,

$$V_i(x,y) = V_i(x, y+1) - D_{n+1}V_i(x, y+\theta_i), \qquad 0 < \theta_i(x,y) < 1. \tag{9}$$

Lemma 4. *Let $F = (U, V_1, ..., V_n) \in H^q$ and let θ_i be as in (9), $i = 0, ..., n$. Then*

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y+\theta_i)| \Big)^q dx \le AC. \tag{10}$$

Proof. We have

$$\sup_{y>0} |V_i(x,y)| \le \sup_{y>0} |V_i(x,y+1)| + \sup_{y>0} |D_{n+1}V_i(x, y+\theta_i)|, \tag{11}$$

$$\sup_{y>0} |D_{n+1}V_i(x, y+\theta_i)| \le 2F^+(x), \tag{12}$$

where $i = 0, ..., n$, $V_0 = U$. Then, (12) implies

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_{n+1}V_i(x, y+\theta_i)| \Big)^q dx \le AC. \tag{13}$$

To get the desired result we apply the Cauchy-Riemann equations $D_i U(x,y) = D_{n+1}V_i(x,y)$. □

In the following result we observe that $\theta_i > 0$ on a set, controlled by estimate (2). Then we use the fact that under conditions (14) the supremum $\sup\limits_{y>0} |F(x,y)|$ is attained at the boundary.

Theorem 5. *Let $0 < p < q$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. Then $F \in h^p$, provided*

$$1)\ M_p(y+1, F) \le C, \qquad 2)\ M_p(y, D_{n+1}U) \le Cy^{-1}, \qquad 3)\ F \in H^q. \tag{14}$$

Moreover, $F \in H^r\, \forall r : p \le r \le q$ if and only if conditions (14) are valid.

Proof. We prove at first that (14) implies $F \in h^p$. Let θ_i be as in (9). Due to 2), (6), and 1), it is enough to show that $\sup\limits_{y>0} |D_{n+1}V_i(\cdot, y+\theta_i(\cdot, y))| \in L^p(\mathbf{R}^n)$. This will follow from the Cauchy-Riemann equations and

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y+\theta_i(x,y))| \Big)^p dx < \infty. \tag{15}$$

Thus, we prove (15). Condition 3) and Lemma 1 imply

$$\sup_{x\in\mathbf{R}^n} |D_i U(x, y+\theta_i)| \le \sup_{x\in\mathbf{R}^n} \frac{AC}{(y+\theta_i)^{1+n/q}} \le AC(y+\alpha_i(y))^{-1-n/q}, \tag{16}$$

where $\alpha_i(y) = \inf\limits_{x\in\mathbf{R}^n} \theta_i(x,y)$. Define

$$L_i = \{x \in \mathbf{R}^n : \sup_{y>0} |D_i U(x, y+\theta_i)| \le \sup_{y>0} \frac{AC}{(y+\alpha_i(y))^{1+n/q}} \le 1\}. \tag{17}$$

Then $\forall x \in CL_i$ (the complement of L_i) we have $\sup\limits_{y>0} |D_i U(x, y + \theta_i)| > 1$. Then

$$(18)\quad \int\limits_{CL_i} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))|\Big)^p dx \le \int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))|\Big)^q dx \le AC$$

due to condition 3) and Lemma 4.

We estimate the integral in (15) over L_i. Observe that for fixed $x \in L_i$,

$$\sup_{y>0} |D_i U(x, y + \theta_i(x, y))| \le \sup_{y>0} |D_i U(x, y + \alpha_i(y))|,$$

and $\alpha_i \equiv \inf\limits_{y>0} \alpha_i(y) > 0$. We put $\gamma = \min\limits_{i=0,\dots,n} \alpha_i > 0$, and take any $0 < y_0 \le \gamma$. Then

$$(19)\quad \int\limits_{L_i} \Big(\sup_{y>0} |D_i U(x, y + \theta_i(x, y))|\Big)^p dx \le \int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |\nabla U(x, y + y_0)|\Big)^p dx \le AC.$$

Indeed, condition 3) and Theorem 3 imply $V_i(x, y) \rightrightarrows^{x}_{y\to\infty} 0$. The same is true for all partial derivatives of V_i, $i = 0, ..., n$, $V_0 = U$. Now the second inequality in (19) follows from 2) and Lemma 3. Taking into account (19), (18), we get (15). Thus, $F \in h^p$.

Now, $F \in H^r\, \forall r : p \le r \le q$, implies (14) by Theorem 3. We prove the converse statement. It is enough to show that $F \in H^p$, or $(\sup\limits_{y>0} |F(\cdot, y)|)^p \in L^1(\mathbf{R}^n)$. Since $F \in h^p$, we apply the Fatou Lemma to have

$$(20)\quad \int\limits_{\mathbf{R}^n} |F(x, 0)|^p dx = \int\limits_{\mathbf{R}^n} \Big(\lim_{y\to 0} |F(x, y)|\Big)^p dx \le \lim_{y\to 0} M_p(y, F) \le C.$$

We claim that $\sup\limits_{y>0} |F(x, y)| = |F(x, 0)|$ and our result follows from (20). Since for fixed $x_0 \in \mathbf{R}^n$ the function $WF(x_0, y) \equiv \sup\limits_{\eta\ge y} |F(x_0, \eta)|$ is nonincreasing in y, we have

$$\sup_{y>0} |F(x_0, y)| = \sup_{y>0} \sup_{\eta\ge y} |F(x_0, \eta)| = \lim_{y\to 0} \sup_{\eta\ge y} |F(x_0, \eta)| = |F(x_0, 0)|.$$

Even if $\sup\limits_{y>0} |F(x_0, y)| = |F(x_0, y_0)|$ for some $y_0 > 0$, then $WF(x_0, y) = WF(x_0, y_0)$ for all $0 \le y \le y_0$, and we may put $|F(x_0, y_0)| = |F(x_0, 0)|$. □

Theorem 6. *Let* $0 < p < q$, $k \in \mathbf{N}$, *and let* $F = (U, V_1, ..., V_n)$ *be a harmonic vector in* $\mathbf{R}^{n+1}_+$. *Then* $F \in H^r\, \forall r : p \le r \le q$ *if and only if*

$$(21)\quad 1)\ M_p(y + 1, F) \le C, \qquad 2)\ M_p(y, D^k_{n+1} U) \le C y^{-k}, \qquad 3)\ F \in H^q.$$

Proof. By Corollary 1, and the inverse statement, proved in [9], conditions (14), (21) are equivalent. □

Theorem 7. *Let* $p > 0$ *and let* $F = (U, V_1, ..., V_n)$ *be a harmonic vector in* $\mathbf{R}^{n+1}_+$ *such that*

$$(22)\quad 1)\ M_p(y + 1, F) \le C, \qquad 2)\ M_p(y, U) \le C, \qquad 3)\ |U(x, y)| \le C.$$

Then $F \in H^r$, $r \ge p$.

Proof. By Theorem 4 we have $F \in H^r$, $r > p$. Let $r = p$. By Lemma 1 and Theorem 3 we have $M_p(y, \triangledown^k U) \leq C y^{-k}$, and we may use Theorem 6. $\square$

Lemma 5. *Let $0 < p < q$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$ such that*

$$\text{(23)} \qquad 1)\, M_p(y+1, F) \leq C, \qquad 2) \int_{\mathbf{R}^n} \Big(\sup_{\eta \geq y} |D_{n+1} U(x, \eta)| \Big)^q dx \leq C.$$

Then $F \in H^q$.

Proof. By the mean-value theorem,

$$\sup_{y>0} |U(x,y)| \leq \sup_{y>0} |U(x, y+1)| + \sup_{y>0} |\nabla U(x,y)|,$$

and it is enough to show that $(\sup_{y>0} |U(\cdot, y+1)|)^p \in L^1$. To prove this, we apply Theorem 3, the mean-value theorem again,

$$|U(x, y+1)| \leq |U(x, y+2)| + \sup_{y>0} |D_{n+1} U(x, y+1+\theta)|,$$

and observe that the conditions of Theorem 7 are satisfied with $y+1$ instead of y. $\square$

4. Proof of Theorem 1.

The proof is given in two lemmata presented below.

Lemma 6. *Let $k \in \mathbf{N}$, $0 < p < q$ and let $F = (U, V_1, ..., V_n)$ be a harmonic vector in $\mathbf{R}^{n+1}_+$. Then F and all its partial derivatives up to order k belong to H^r, $p \leq r \leq q$, if and only if*

$$\text{(24)} \quad 1)\, M_p(y + y_0, F) \leq C(y_0) \ \ \forall y_0 > 0, \qquad 2) \int_{\mathbf{R}^n} \Big(\sup_{\eta \geq y} |D^k_{n+1} U(x, \eta)| \Big)^q dx \leq C.$$

Proof. The **only if** part is trivial. We prove **if** by induction. Let $k = 1$. We show at first that $F \in H^r$, $p \leq r \leq q$. By Lemma 5 we have $F \in H^q$, and it is enough to show that $F \in H^p$. To this end, we apply the mean-value theorem and repeat the proof of Theorem 5 begining with (15). As in Theorem 5 we define L_i, (see (17)), and (18) follows from 2). The last estimate in (19) follows from Theorem 3 and Lemma 7. Conditions (22) are satisfied with ∇U instead of F, $D_{n+1} U$ instead of U, and $y + y_0$ instead of y.

To show that all partial derivatives of the first order belong to H^r, $p \leq r \leq q$ one has to proceed as above by changing ∇U by $\nabla^2 U$, and $D^2_{n+1} U$ by $D_{n+1} U$.

Assume that the statement is true for $k - 1$, and we have to prove it for k. By Theorem 3 we have 1) with $D^{k-1}_{n+1} U$ instead of F, and the result follows. $\square$

Lemma 7. *Conditions of the theorem are equivalent to conditions of the previous lemma.*

Proof. It is enough to prove that $M_p(y+1,F) \le C$ and 2) of (24) imply 1) of (24). This will follow from $F \in H^q$. Since $V_i(x,y) \rightrightarrows^x_{y\to\infty} 0$, $i=1,...,n$, it is enough to show that $U \in H^q$. We will subsequently show that all $D^{k-1}_{n+1}U, D^{k-2}_{n+1}U, ..., U \in H^q$. In fact, we prove that $D^{k-1}_{n+1}U \in H^q$. The proof of $D^{k-2}_{n+1}U, ..., U \in H^q$ is similar.

Observe that $D^{k-1}_{n+1}U \in H^q$ follows from $D^{k-1}_{n+1}U \in h^q$. Indeed, let $D^{k-1}_{n+1}U \in h^q$. By the mean-value theorem we have

$$\sup_{y>0} |D^{k-1}_{n+1}U(x,y)| \le \sup_{y>0} |D^{k-1}_{n+1}U(x,y+1)| + \sup_{y>0} |D^{k}_{n+1}U(x,y)|,$$

and we may apply Lemma 7, (with $p=q$, $D^{k-1}_{n+1}U(x,y+1)$ instead of $U(x,y)$, $\nabla^{k-1}F$ instead of F), to obtain

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |D^{k-1}_{n+1}U(x,y+1)| \Big)^p dx < \infty.$$

The assumption $D^{k-1}_{n+1}U \in h^q$ and Lemma 1 are used to satisfy the third condition of Lemma 7. The above inequality gives $D^{k-1}_{n+1}U \in H^q$.

Thus, it remains to prove that $D^{k-1}_{n+1}U(x,y) \in h^q$. By the mean-value theorem,

$$|D^{k-1}_{n+1}U(x,y)| \le |D^{k-1}_{n+1}U(x,y+1)| + \sup_{y>0} |D^{k}_{n+1}U(x,y)|,$$

and it is enough to prove that $D^{k-1}_{n+1}U(x,y+1) \in h^q$. Again, by the mean-value theorem,

$$|D^{k-1}_{n+1}U(x,y+1)| \le |D^{k-1}_{n+1}U(x,y+1+1)| + \sup_{y>0} |D^{k}_{n+1}U(x,y)|,$$

but now we may use the assumption $M_p(y+1,F) \le C$ to show that for $r \ge p$, $M_r(y+2, D^{k-1}_{n+1}U) \le C$. To this end, we apply Theorem 3, Lemma 1, and take $y+1$ instead of y. □

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