SOME PROPERTIES OF CONJUGATE HARMONIC FUNCTIONS IN A HALF-SPACE

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Abstract. We prove a multi-dimensional analog of the Theorem of Hardy and Littlewood about the logarithmic bound of the $L^p$-average of the conjugate harmonic functions, $0 < p \leq 1$. We also give sufficient conditions for a harmonic vector to belong to $H^p(R^{n+1}_+)$, $0 < p \leq 1$.

1. Introduction and statements of main results

The following result of Hardy and Littlewood [6] is classical.

Theorem 1. Let $0 < p \leq 1$, and let $f(z) = u(z) + iv(z)$ be an analytic function in the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$, such that

$$1) \quad v(0) = 0,$$

$$2) \quad M_p(r, u) := \left(\frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta}|^p \, d\theta\right)^{1/p} \leq C, \quad 0 \leq r < 1.$$  

Then

$$M_p(r, v) \leq AC + AC \left(\log \frac{1}{1 - r}\right)^{1/p}.$$  

In this paper we prove an analog of Theorem 1 for conjugate harmonic functions in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. The case $p < (n - 1)/n$ leads to additional difficulties, since $|F|^p$ is subharmonic, provided $p \geq (n - 1)/n$, [15]. We refer the reader to the classical works [3], [15], [17], [5], [2], [4], [18] for the history and different results related to the classes $S^p(\mathbb{R}^{n+1}_+), h^p(\mathbb{R}^{n+1}_+), H^p(\mathbb{R}^{n+1}_+)$, (all definitions are given in Section 2).

We have

Theorem 2. Let $0 < p \leq 1$, and let $F$,

$$F(x, y) = (U(x, y), V_1(x, y), V_2(x, y), \ldots, V_n(x, y)), \quad (x, y) \in \mathbb{R}^{n+1}_+,$$

be the harmonic vector such that

$$1) \quad V_i \Rightarrow_{y \to \infty}^x 0, \quad i = 1, \ldots, n, \quad 2) \quad M_p(y, U) \leq C, \quad 3) \quad M_p(1, F) \leq C.$$  

Then

$$M_p(y, V) \leq AC + AC |\log y|^{1/p}.$$
The third condition in (3) appears after the application of the Main Theorem of calculus, see (8), (9). The logarithmic bound comes from the estimate
\[ \int_{\mathbb{R}^n} \left( \sup_{\xi \geq t} |\nabla U(x, \xi)| \right)^p \, dx \leq ACt^{-p} \]
in the integral
\[ \int_0^1 t^{p-1} dt \int_{\mathbb{R}^n} \left( \sup_{\xi \geq t} |\nabla U(x, \xi)| \right)^p \, dx, \]
see Lemmata 3, 4, 5.

To control the logarithmic blow up, we use the "Littlewood-Paley"-type condition:
\[ I(p) := \int_0^1 t^{p-1} dt \int_{\mathbb{R}^n} \left( \sup_{\xi \geq t} |\nabla U(x, \xi)| \right)^p \, dx < \infty. \]

Our second result is

**Theorem 3.** Let \(0 < p \leq 1\), and let \(F = (U, V_1, ..., V_n)\) be the harmonic vector such that
(5) 1) \(F \rightarrow \xi \rightarrow \infty 0\), 2) \(M_p(1, F) \leq C\), 3) \(I(p) < \infty\).

Then \(F \in H^p\).

It is unlikely that \(F \in H^p\) implies \(I(p) < \infty\).

The paper is organized as follows. In section 2 we give all necessary definitions and auxiliary results used in the sequel. In Section 3 and 4 we prove Theorems 2 and 3. For convenience of the reader we split our proofs into elementary Lemmata.

2. Auxiliary results

Let \(U(x, y)\) be a harmonic function in \(\mathbb{R}_{++}^{n+1} = \mathbb{R}^n \times (0, \infty)\). We say that the vector-function \(V(x, y) = (V_1(x, y), ..., V_n(x, y))\) is the conjugate of \(U(x, y)\) in the sense of M. Riesz [14], [16], if \(V_k(x, y), k = 1, ..., n\) are harmonic functions, satisfying the generalized Cauchy-Riemann conditions:
\[ \frac{\partial U}{\partial y} + \sum_{k=1}^n \frac{\partial V_k}{\partial x_k} = 0, \quad \frac{\partial V_i}{\partial x_k} = \frac{\partial V_k}{\partial x_i}, \quad \frac{\partial U}{\partial x_i} = \frac{\partial V_i}{\partial y}, \quad i \neq k, \ k = 1, ..., n. \]

If \(U(x, y)\) and \(V(x, y)\) are conjugate in \(\mathbb{R}_{++}^{n+1}\) in the above sense, then the vector-function
\[ F(x, y) = (U(x, y), V(x, y)) = (U(x, y), V_1(x, y), ..., V_n(x, y)) \]
is called a harmonic vector.

Define
\[ M_p(y, F) = \left( \int_{\mathbb{R}^n} \left( U^2(x, y) + \sum_{i=1}^n V_i^2(x, y) \right)^{p/2} \, dx \right)^{1/p}, \quad p > 0. \]

Now we define the space \(H^p(\mathbb{R}_{++}^{n+1})\). We follow the work of Fefferman and Stein[4]. Let \(U(x, y)\) be a harmonic function in \(\mathbb{R}_{++}^{n+1}\), and let \(U_{j_1j_2j_3...j_k}\) denote a component
of a symmetric tensor of rank \( k \), \( 0 \leq j_i \leq n \), \( i = 1, ..., n \). Suppose also that the trace of our tensor is zero, meaning
\[
\sum_{j=0}^{n} U_{j j_3 ... j_k} (x, y) = 0, \quad \forall j_3, ..., j_k.
\]
The tensor of rank \( k + 1 \) can be obtained from the above tensor of rank \( k \) by passing to its gradient:
\[
U_{j_1 j_2 ... j_k j_{k+1}} (x, y) = \frac{\partial}{\partial x_{j_{k+1}}} (U_{j_1 j_2 j_3 ... j_k} (x, y)), \quad x_0 = y, \quad 0 \leq j_{k+1} \leq n.
\]

**Definition ([4]).** We say that \( U \in H^p(\mathbb{R}_{+}^{n+1}), p > 0 \), if there exists a tensor of rank \( k \) of the above type with the properties:
\[
U_{0 ... 0} (x, y) = U (x, y), \quad \sup_{y > 0} \int_{\mathbb{R}^n} \left( \sum_{(j)} U_{j}^2 (x, y) \right)^{p/2} dx < \infty, \quad (j) = (j_1, ..., j_k).
\]

It is well-known that the function \( \left( \sum_{(j)} U_{j}^2 (x, y) \right)^{p/2} \) is subharmonic for \( p \geq p_k = (n - k)/(n + k - 1) \), see [3], [4], [16].

We remind that the radial and the non-tangential maximal functions are defined as follows:
\[
F^+ (x) = \sup_{y > 0} |F (x, y)|, \quad N_{\alpha} (F) (x^0) = \sup_{(x,y) \in \Gamma_{\alpha} (x^0)} |F (x, y)|.
\]

Here
\[
\Gamma_{\alpha} (x^0) = \{(x, y) \in \mathbb{R}_{+}^{n+1} : |x - x^0| < \alpha y\}, \quad \alpha > 0,
\]
is an infinite cone with the vertex at \( x^0 \). It is well-known [4] that
\[
F \in H^p(\mathbb{R}_{+}^{n+1}) \iff N_{\alpha} (F) \in L^p \iff F^+ \in L^p, \quad p > 0.
\]

We also define the **weak maximal function**
\[
WF (x, y) = \sup_{\zeta \geq y} |F (x, \zeta)|, \quad y > 0.
\]
The above expression is understood as follows: we fix \( x \), and for fixed \( y \) we find the supremum over all \( \zeta \geq y \).

We will use the following results.

**Lemma 1.** ([4], p.173). Suppose \( w \) is harmonic in \( \mathbb{R}_{+}^{n+1} \), and \( M_p (y, u) \leq C \) for some \( p, 0 < p < \infty \). Then
\[
(6) \quad \sup_{x \in \mathbb{R}^n} |u(x, y)| \leq A y^{-n/p}, \quad 0 < y < \infty.
\]

**Theorem 4.** ([5], p.267). Let \( 0 < p \leq 1, a > 0 \), let \( w : \mathbb{R}_{+}^{n+1} \to [0, \infty) \) be a function such that \( w^p \) is subharmonic and satisfies
\[
J_{a,p} := \int_{\mathbb{R}_{+}^{n+1}} t^{ap-1} w(x,t)^p dx dt < +\infty,
\]
and for each \((x, t) \in \mathbb{R}^n \times [0, +\infty)\) let

\[
w_a(x, t) := \frac{1}{\Gamma(a)} \int_0^{\infty} s^{a-1} w(x, s + t) \, ds.
\]

Then \(w_a\) is subharmonic on \(\mathbb{R}^{n+1}_+\) and is finite a.e. on \(\mathbb{R}^n\), and for all \(t \geq 0\),

\[
\int_{\mathbb{R}^n} w(x, t)^p \, dx \leq AC(a, n, p) J_{a,p}.
\]

**Theorem 5.** ([5], p.269). Let \(m \in \mathbb{N}, p \geq (n-1)/(m+n-1)\) (if \(n = 1\) we suppose \(p > 0\)), and let \(u : \mathbb{R}_+^n \rightarrow \mathbb{R}\) be harmonic. Then, for all \(t > 0\),

\[
\int_{\mathbb{R}^n} |\nabla^m u(x, t)|^p \, dx \leq A(m, n, p)t^{-mp-1}\int_{t/2}^{3t/2} ds \int R^n |u(x, s)|^p \, dx.
\]

**Lemma 2.** ([13], p.2464). Let \(p > 0\) and let \(F = (U, V_1, ..., V_n)\) satisfy \(M_p(y, U) \leq C\). Then

\[
M_p(y, \nabla^k F) \leq ACy^{-k}, \quad k \in \mathbb{N}.
\]

**Notation.** We denote by \(D^k_i f(x, y)\) the partial derivative of the function \(f\) of the order \(k\) with respect to \(x_i, i = 1, 2, ..., n+1\). The notation \(f(x, y) \rightarrow x^{y \rightarrow \infty} 0\) means that \(f(x, y)\) converges to 0 uniformly with respect to \(x\), provided \(y \rightarrow \infty\), \(\nabla^k f(x) = (\frac{\partial^k f(x)}{\partial x^1}, ..., \frac{\partial^k f(x)}{\partial x^n})\). Everywhere below the constants \(A(k, n), C, K\) depend only on the parameters pointed in parentheses, and may be different from time to time.

### 3. Proof of Theorem 2.

**Lemma 3.** Let \(p > 0\), and let \(F = (U, V_1, ..., V_n)\) satisfy \(M_p(1, F) \leq C\). Then

\[
(7) \quad M_p(y, V_i) \leq AC + \int_{\mathbb{R}^n} \left( \int_y^{1} \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt \right)^p \, dx, \quad i = 0, 1, ..., n, \quad V_0 = U.
\]

**Proof.** By the Main Theorem of Calculus, and the Cauchy-Riemann equations, we have

\[
(8) \quad V_i(x, y) - V_i(x, 1) = -\int_y^1 \frac{\partial V_i(x, t)}{\partial t} \, dt = -\int_y^1 \frac{\partial U(x, t)}{\partial x_i} \, dt,
\]

\(i = 1, 2, ..., n,\)

\[
(9) \quad U(x, y) - U(x, 1) = -\int_y^1 \frac{\partial U(x, t)}{\partial t} \, dt.
\]

Then,

\[
M_p(y, V_i) \leq M_p(1, V_i) + \int_{\mathbb{R}^n} \left( \int_y^1 \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt \right)^p \, dx,
\]

and the result follows. \(\square\)
Lemma 4. Let \( p > 0 \) and let \( 0 < y < 1 \). Then

\[
\int_{\mathbb{R}^n} \left( \int_y^1 \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt \right)^p \, dx \leq 
\]

\[
y^p \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\nabla U(x, \xi)| \right)^p \, dx + 2p \int_y^1 t^{p-1} \, dt \int_{\mathbb{R}^n} \left( \sup_{\xi \geq t} |\nabla U(x, \xi)| \right)^p \, dx.
\]

Proof. Denote

\[
\Psi(x, y) := \int_y^1 \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt.
\]

Following [6] consider

\[
\Phi(x, y) := \Psi(x, y)^p - y^p \left( - \frac{\partial \Psi(x, y)}{\partial y} \right)^p, \quad \Omega(y) := \{ x \in \mathbb{R}^n : \Phi(x, y) > 0 \}.
\]

By definition of \( \Omega(y) \),

\[
\int_{\mathbb{R}^n \setminus \Omega(y)} \Psi(x, y)^p \, dx \leq y^p \int_{\mathbb{R}^n \setminus \Omega(y)} \left( - \frac{\partial \Psi(x, y)}{\partial y} \right)^p \, dx.
\]

Next, the reasons which are similar to those in [6], imply

\[
\int_{\Omega(y)} \Phi(x, y) \, dx - \int_{\Omega(a)} \Phi(x, a) \, dx = \int_y^a \, dx \int_{\Omega(\xi)} - \frac{\partial \Phi(x, \xi)}{\partial \xi} \, dx, \quad 0 < y < a \leq 1.
\]

Moreover, using \( \frac{\partial^2 \Psi}{\partial \xi^2} \geq 0 \) for almost every \( 0 < \xi < 1 \), we have

\[
- \frac{\partial \Phi}{\partial \xi} = -p \Psi^{p-1} \frac{\partial \Psi}{\partial \xi} + p \xi^{p-1} \left( - \frac{\partial \Psi}{\partial \xi} \right)^p - p \xi^p \left( - \frac{\partial \Psi}{\partial \xi} \right)^{p-1} \frac{\partial^2 \Psi}{\partial \xi^2} \leq 
\]

\[
p \left( - \frac{\partial \Psi}{\partial \xi} \right) \left( \Psi^{p-1} + \xi^{p-1} \left( - \frac{\partial \Psi}{\partial \xi} \right)^{p-1} \right) \leq 2p \xi^{p-1} \left( - \frac{\partial \Psi}{\partial \xi} \right)^p.
\]

Here the last inequality follows from the definition of \( \Omega(\xi) \) and \( 0 < p < 1 \). Since

\[
\Phi(x, y) \leq (1 - y) \left| \frac{\partial \Psi(x, y)}{\partial y} \right|,
\]

the function \( \Phi(x, y) \) is negative, provided \( y \) is sufficiently close to 1, and we can take \( a \) such that \( \Omega(a) = \emptyset \). Hence, (12) yields

\[
\int_{\Omega(y)} \Phi(x, y) \, dx \leq 2p \int_y^a \xi^{p-1} \, dx \int_{\Omega(\xi)} \left( - \frac{\partial \Phi(x, \xi)}{\partial \xi} (x, \xi) \right)^p \, dx \leq 
\]

\[
2p \int_y^1 \xi^{p-1} \, dx \int_{\mathbb{R}^n} \left( - \frac{\partial \Phi(x, \xi)}{\partial \xi} (x, \xi) \right)^p \, dx.
\]

Adding

\[
\int_{\Omega(y)} \Psi(x, y)^p \, dx \leq \int_{\Omega(y)} \Phi(x, y) \, dx + y^p \int_{\Omega(y)} \left( - \frac{\partial \Phi(x, y)}{\partial y} \right)^p \, dx
\]

with (11), and using (13), we obtain (10). □
The next result is crucial.

**Lemma 5.** Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be such that

\begin{align}
1) \quad V_i &\Rightarrow_{y \to \infty} 0, \quad i = 1, ..., n, \\
2) \quad M_p(y, U) &\leq C.
\end{align}

Then

\begin{align}
\left( \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\phi_{ij}(x, \xi)| \right)^p dx \right)^{1/p} &\leq ACy^{-1},
\end{align}

where $\phi_{ij}(x, y)$ is a coordinate of $\nabla V_i(x, y)$, $j = 1, ..., n + 1$, $x_{n+1} = y$, $i = 0, ..., n$, $V_0 = U$.

**Proof.** Fix $p > 0$ and let $l = \inf \{ j \in \mathbb{N} : p > p_j := (n-1)/(j + n - 1) \}$. Since $\nabla V_i(x, y) \Rightarrow_{y \to \infty} 0$, we may use the following relation (see [5] or [4])

\[ \phi_{ij}(x, y) = \frac{1}{(2l - 2)!} \int_y^\infty (s - y)^{2l-2} D_{n+1}^{2l-1} \phi_{ij}(x, s) ds = \]

\[ \frac{1}{(2l - 2)!} \int_0^\infty s^{2l-2} D_{n+1}^{2l-1} \phi_{ij}(x, s + y) ds. \]

We have

\[ \sup_{\xi \geq y} |\phi_{ij}(x, \xi)| \leq R(x, y), \]

where

\[ R(x, y) := \frac{1}{(2l - 2)!} \int_0^\infty s^{2l-2} \left( \sup_{\xi \geq y} |\nabla \nabla_{n+1}^{l-1} \phi_{ij}(x, s + \xi)| \right) ds. \]

To prove (15) it is enough to show that

\begin{align}
\left( \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\phi_{ij}(x, s + \xi)| \right)^p dx \right)^{1/p} &\leq AC y^{-1}.
\end{align}

Since

\[ \left( |\nabla \nabla_{n+1}^{l-1} \phi_{ij}(x, \xi)| \right)^p \]

is subharmonic [3], the function

\[ w^p(x, s + y) := \left( \sup_{\xi \geq y} |\nabla \nabla_{n+1}^{l-1} \phi_{ij}(x, s + \xi)| \right)^p \]

is also subharmonic, and we may apply Theorem 4 (take $a = 2l - 1, A = A(l, n, p)$) to obtain

\[ \int_{\mathbb{R}^n} |R(x, y)|^p dx \leq A \int_0^\infty s^{(2l-1)p-1} ds \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\nabla \nabla_{n+1}^{l-1} \phi_{ij}(x, s + \xi)| \right)^p dx = \]

\[ A \int_0^\infty s^{(2l-1)p-1} ds \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\nabla \nabla_{n+1}^{l-1} \phi_{ij}(x, s + \xi)|^{p_j} \right)^{p/p_j} dx. \]
By the choice of $p_j$, we have $p/p_j > 1$ and we may use the well-known $L^{p/p_j}$-boundedness of the maximal operator:

$$\int_{\mathbb{R}^n} |R(x, y)|^p dx \leq A \int_0^\infty s^{(2l-1)p-1} ds \int_{\mathbb{R}^n} |\nabla^l D_{n+1}^- \phi_{ij}(x, s + y)|^p dx.$$

Since $D_{n+1}^- \phi_{ij}(x, y)$ is the $l$-th derivative of $V_i$, we use Lemma 2 to get

$$\int_{\mathbb{R}^n} |\nabla^l D_{n+1}^- \phi_{ij}(x, y)|^p dx \leq \int_{\mathbb{R}^n} |\nabla^{2l} F(x, y)|^p dx \leq C y^{−2lp}.$$

This gives

$$\int_{\mathbb{R}^n} |R(x, y)|^p dx \leq A(l, n, p) C \int_0^\infty s^{(2l-1)p-1}(s + y)^{−2lp} ds = A(l, n, p) Cy^{−p},$$

and (16) is proved. $\square$

**Proof of Theorem 2.** The proof follows from Lemmata 3, 4, 5.

4. **Proof of Theorem 3.**

**Lemma 6.** Let $p > 0$. Then $F = (U, V_1, \ldots, V_n) \in H^p$ iff

1) $F \Rightarrow_{y \to \infty} 0$, 2) $\left( \sup_{y > 0} |U(x, \eta)| \right)^p dx < C$.

**Proof.** Let $F \in H^p$, then both 1) and 2) are well-known, [4]. We prove the converse in two steps. At first we show that

$$\left( \sup_{y > 0} |U(x, y)| \right)^p dx \leq C.$$  

Then we prove that (18) implies

$$\left( \sup_{y > 0} |V_i(\cdot, y)| \right)^p \in L^1(\mathbb{R}^n), \quad i = 1, \ldots, n.$$

To prove (18), we observe that

$$\sup_{y > 0} |U(x, y)| = \sup_{y > 0} \sup_{\eta \geq y} |U(x, \eta)| = \lim_{y \to 0} \sup_{\eta \geq y} |U(x, \eta)|.$$

Hence, using 2) and Fatou’s Lemma, we obtain

$$\int_{\mathbb{R}^n} \left( \sup_{y > 0} |U(x, y)| \right)^p dx \leq \lim_{y \to 0} \int_{\mathbb{R}^n} \left( \sup_{\eta \geq y} |U(x, \eta)| \right)^p dx \leq C.$$

It remains to show (19). Using Cauchy-Riemann equations, we have

$$V_i(x, y) = [V_i]_{0 \ldots 0}(x, y) = \frac{(-1)^k}{(k - 1)!} \int_0^\infty s^{k-1} D_{n+1}^k V_i(x, s + y) ds =$$
\[ \frac{(-1)^k}{(k-1)!} \int_0^\infty s^{k-1}D_{n+1}^kU(x, s + y)ds, \]

where \( k \) is chosen such that the function \( \left( \sum_{(j)} U^2_{(j)}(x, y) \right)^{p/2} \) is subharmonic, \( p \geq p_k = (n-k)/(n+k-1) \), see \([3],[4],[16]\). Since the expression in (20) is one of the tensor coordinates of \( U_{(j)} \), see \([4]\), page 169), we have

\[
\int_{\mathbb{R}^n} \left( \sup_{y>0} |V_i(x, y)| \right)^p dx \leq \int_{\mathbb{R}^n} \left( \sup_{y>0} \sum_{(j)} U^2_{(j)}(x, y) \right)^{p/2} dx = \sup_{y>0} \int_{\mathbb{R}^n} \left( \sum_{(j)} U^2_{(j)}(x, y) \right)^{p/2} dx \leq C \int_{\mathbb{R}^n} \left( \sup_{y>0} |U(x, y)| \right)^p dx.
\]

The last estimate is proved in \([4]\), page 170.

Lemma 7. Let \( p > 0 \) and let \( F = (U, V_1, ..., V_n) \) be such that \( V_i \Rightarrow_{y \to \infty} 0, \ i = 1, ..., n \). Then \( F \in H^p \), provided

\[ \begin{align*}
1) & \ M_p(1, F) \leq C, \\
2) & \ M_p(y, U) \leq C, \\
3) & \ I(p) < \infty.
\end{align*} \]

Proof. At first we prove that

\[ 2^{-p} \int_G \left( \sup_{y>0} |V_i(x, y)| \right)^p dx \leq AC + \int_G \left( \sup_{y>0} |\nabla U(x, y)| \right)^p dx, \quad \forall G \subset \mathbb{R}^n. \tag{22} \]

This follows from \((8), (9)\) the inequality

\[ 2^{-p} \left( \sup_{y>0} |V_i(x, y)| \right)^p \leq \left( |V_i(x, 1)| \right)^p + \left( \sup_{y>0} |\nabla U(x, y)| \right)^p, \]

\( j = 0, 1, 2, ..., n, V_0 = U \), and the first condition in (21).

Now we finish the proof. Assume that the lemma is not true. Then \( \forall N > 0 \) there exists a set \( E \subset \mathbb{R}^n, 0 < m(E) < \infty, \) such that

\[ \int_E \left( \sup_{y>0} |V_i(x, y)| \right)^p dx \geq 4N, \]

for some \( i = 0, 1, ..., n \), \( V_0 = U \). Hence, taking \( G = E \) in (22) we have

\[ \int_E \left( \sup_{y>0} |\nabla U(x, y)| \right)^p dx \geq 2N. \]

Then

\[ \sup_{y>0} |\nabla U(x, y)| = \sup_{y>0} \sup_{\xi \geq y} |\nabla U(x, \xi)| = \lim_{y \to 0} \sup_{\xi \geq y} |\nabla U(x, \xi)| \]

and Lemma Fatou imply

\[ \lim_{y \to 0} \int_E \left( \sup_{\xi \geq y} |\nabla U(x, \xi)| \right)^p dx \geq \int_E \left( \sup_{y>0} |\nabla U(x, y)| \right)^p dx \geq 2N. \]
But this contradicts the third condition of the lemma, since
\[ I(p) \geq \int_0^1 dy \int_{\mathbb{R}^n} \left( \sup_{\xi \geq y} |\nabla U(x, \xi)| \right)^p dx \geq \int_0^1 dy \int_E \left( \sup_{\xi \geq y} |\nabla U(x, \xi)| \right)^p dx \geq N. \]

Lemma 8. Let \( p > 0 \) and let \( F = (U, V_1, \ldots, V_n) \) be the harmonic vector satisfying conditions of Theorem 3. Then (18) holds.

Proof. We argue as in the previous Lemma. We have (9), and
\[ 2^{-p} \int_{\mathbb{R}^n} \left( \sup_{y>0} |U(x,y)| \right)^p dx \leq \int_{\mathbb{R}^n} |U(x,1)|^p dx + \int_{\mathbb{R}^n} \left( \sup_{y>0} |\nabla U(x,y)| \right)^p dx \]
leads to a contradiction.

Proof of Theorem 3. The proof follows from Lemmata 8, 7 and 6.

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