SOME PROPERTIES OF CONJUGATE HARMONIC FUNCTIONS IN A HALF-SPACE

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Abstract. We prove a multi-dimensional analog of the Theorem of Hardy and Littlewood about the logarithmic bound of the L^p- average of the conjugate harmonic functions, $0 < p \leq 1$. We also give sufficient conditions for a harmonic vector to belong to $H^p(\mathbf{R}^{n+1}_+)$, $0 < p \leq 1$.

1. Introduction and statements of main results

The following result of Hardy and Littlewood [6] is classical.

Theorem 1. *Let $0 < p \leq 1$, and let $f(z) = u(z) + iv(z)$ be an analytic function in the unit disc $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, such that*

$$(1) \qquad 1)\ v(0) = 0, \qquad 2)\ M_p(r,u) := \Big(\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p d\theta\Big)^{1/p} \leq C,\ \ 0 \leq r < 1.$$

Then

$$(2) \qquad M_p(r,v) \leq AC + AC\Big(\log\frac{1}{1-r}\Big)^{1/p}.$$

In this paper we prove an analog of Theorem 1 for conjugate harmonic functions in $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times (0,\infty)$. The case $p < (n-1)/n$ leads to additional difficulties, since $|F|^p$ is subharmonic, provided $p \geq (n-1)/n$, [15]. We refer the reader to the classical works [3], [15], [17], [5], [2], [4], [18] for the history and different results related to the classes $S^p(\mathbf{R}^{n+1}_+), h^p(\mathbf{R}^{n+1}_+)$, $H^p(\mathbf{R}^{n+1}_+)$, (all definitions are given in Section 2).

We have

Theorem 2. *Let $0 < p \leq 1$, and let F,*

$$F(x,y) = (U(x,y), V_1(x,y), V_2(x,y), ..., V_n(x,y)), \qquad (x,y) \in \mathbf{R}^{n+1}_+,$$

be the harmonic vector such that

$$(3) \qquad 1)\ V_i \rightrightarrows^x_{y\to\infty} 0,\ i = 1,...,n, \quad 2)\ M_p(y,U) \leq C, \qquad 3)\ M_p(1,F) \leq C.$$

Then

$$(4) \qquad M_p(y,V) \leq AC + AC\,|\log y|^{1/p}.$$

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The third condition in (3) appears after the application of the Main Theorem of calculus, see (8), (9). The logarithmic bound comes from the estimate

$$\int_{\mathbf{R}^n} \Big(\sup_{\xi \geq t} |\nabla U(x,\xi)| \Big)^p dx \leq ACt^{-p}$$

in the integral

$$\int_y^1 t^{p-1} dt \int_{\mathbf{R}^n} \Big(\sup_{\xi \geq t} |\nabla U(x,\xi)| \Big)^p dx,$$

see Lemmata 3, 4, 5.

To control the logarithmic blow up, we use the "Littlewood-Paley"- type condition:

$$I(p) := \int_0^1 t^{p-1} dt \int_{\mathbf{R}^n} \Big(\sup_{\xi \geq t} |\nabla U(x,\xi)| \Big)^p dx < \infty.$$

Our second result is

Theorem 3. *Let $0 < p \leq 1$, and let $F = (U, V_1, ..., V_n)$ be the harmonic vector such that*

(5) $$1)\ F \rightrightarrows^x_{y\to\infty} 0, \qquad 2)\ M_p(1,F) \leq C, \qquad 3)\ I(p) < \infty.$$

Then $F \in H^p$.

It is unlikely that $F \in H^p$ implies $I(p) < \infty$.

The paper is organized as follows. In section 2 we give all necessary definitions and auxiliary results used in the sequel. In Section 3 and 4 we prove Theorems 2 and 3. For convenience of the reader we split our proofs into elementary Lemmata.

2. Auxiliary results

Let $U(x,y)$ be a harmonic function in $\mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times (0,\infty)$. We say that the vector-function $V(x,y) = (V_1(x,y), ..., V_n(x,y))$ is the conjugate of $U(x,y)$ in the sense of M. Riesz [14], [16], if $V_k(x,y)$, $k = 1, ..., n$ are harmonic functions, satisfying the generalized Cauchy-Riemann conditions:

$$\frac{\partial U}{\partial y} + \sum_{k=1}^n \frac{\partial V_k}{\partial x_k} = 0, \qquad \frac{\partial V_i}{\partial x_k} = \frac{\partial V_k}{\partial x_i}, \qquad \frac{\partial U}{\partial x_i} = \frac{\partial V_i}{\partial y}, \qquad i \neq k,\ k = 1, ..., n.$$

If $U(x,y)$ and $V(x,y)$ are conjugate in $\mathbf{R}^{n+1}_+$ in the above sense, then the vector-function

$$F(x,y) = (U(x,y), V(x,y)) = (U(x,y), V_1(x,y), ..., V_n(x,y))$$

is called a harmonic vector.

Define

$$M_p(y,F) = \Big(\int_{\mathbf{R}^n} \Big(U^2(x,y) + \sum_{i=1}^n V_i^2(x,y) \Big)^{p/2} dx \Big)^{1/p}, \qquad p > 0.$$

Now we define the space $H^p(\mathbf{R}^{n+1}_+)$. We follow the work of Fefferman and Stein[4]. Let $U(x,y)$ be a harmonic function in $\mathbf{R}^{n+1}_+$, and let $U_{j_1 j_2 j_3 \cdots j_k}$ denote a component of a symmetric tensor of rank k, $0 \le j_i \le n$, $i = 1, ..., n$. Suppose also that the trace of our tensor is zero, meaning

$$\sum_{j=0}^{n} U_{jjj_3\dots j_k}(x,y) = 0, \qquad \forall j_3, ..., j_k.$$

The tensor of rank $k+1$ can be obtained from the above tensor of rank k by passing to its gradient:

$$U_{j_1 j_2 \dots j_k j_{k+1}}(x,y) = \frac{\partial}{\partial x_{j_{k+1}}}(U_{j_1 j_2 j_3 \dots j_k}(x,y)), \qquad x_0 = y,\ 0 \le j_{k+1} \le n.$$

Definition ([4]). *We say that $U \in H^p(\mathbf{R}^{n+1}_+)$, $p > 0$, if there exists a tensor of rank k of the above type with the properties:*

$$U_{0\dots0}(x,y) = U(x,y), \qquad \sup_{y>0} \int_{\mathbf{R}^n} \Big(\sum_{(j)} U^2_{(j)}(x,y) \Big)^{p/2} dx < \infty, \qquad (j) = (j_1, ...j_k).$$

It is well-known that the function $\Big(\sum_{(j)} U^2_{(j)}(x,y) \Big)^{p/2}$ is subharmonic for $p \ge p_k = (n-k)/(n+k-1)$, see [3],[4],[16].

We remind that the *radial* and the *non-tangential* maximal functions are defined as follows:

$$F^+(x) = \sup_{y>0} |F(x,y)|, \qquad N_\alpha(F)(x^0) = \sup_{(x,y)\in\Gamma_\alpha(x^0)} |F(x,y)|.$$

Here

$$\Gamma_\alpha(x^0) = \{(x,y) \in \mathbf{R}^{n+1}_+ : |x - x^0| < \alpha y\}, \qquad \alpha > 0,$$

is an infinite cone with the vertex at x^0. It is well-known [4] that

$$F \in H^p(\mathbf{R}^{n+1}_+) \iff N_\alpha(F) \in L^p \iff F^+ \in L^p,\ p > 0.$$

We also define the *weak maximal function*

$$WF(x,y) = \sup_{\zeta \ge y} |F(x,\zeta)|, \qquad y > 0.$$

The above expression is understood as folows: we fix x, and for fixed y we find the supremum over all $\zeta \ge y$.

We will use the following results.

Lemma 1. ([4], *p.*173). *Suppose w is harmonic in $\mathbf{R}^{n+1}_+$, and $M_p(y,u) \le C$ for some p, $0 < p < \infty$. Then*

$$\sup_{x\in\mathbf{R}^n} |u(x,y)| \le A y^{-n/p}, \qquad 0 < y < \infty. \tag{6}$$

Theorem 4. ([5], *p.*267). *Let $0 < p \le 1$, $a > 0$, let $w : \mathbf{R}^{n+1}_+ \to [0,\infty)$ be a function such that w^p is subharmonic and satisfies*

$$J_{a,p} := \int_{\mathbf{R}^{n+1}_+} t^{ap-1} w(x,t)^p dx dt < +\infty,$$

and for each $(x,t)\in \mathbf{R}^n\times[0,+\infty)$ *let*

$$w_a(x,t):=\frac{1}{\Gamma(a)}\int\limits_0^{+\infty} s^{a-1}w(x,s+t)\,ds.$$

Then w_a *is subharmonic on* $\mathbf{R}^{n+1}_+$ *and is finite a.e. on* $\mathbf{R}^n$, *and for all* $t\ge 0$,

$$\int\limits_{\mathbf{R}^n} w(x,t)^p\,dx\le AC(a,n,p)\,J_{a,p}.$$

Theorem 5. ([5], *p.269*). *Let* $m\in\mathbf{N}$, $p\ge(n-1)/(m+n-1)$ *(if* $n=1$ *we suppose* $p>0$*), and let* $u:\mathbf{R}^{n+1}_+\to\mathbf{R}$ *be harmonic. Then, for all* $t>0$,

$$\int\limits_{\mathbf{R}^n}|\nabla^m u(x,t)|^p dx\le A(m,n,p)t^{-mp-1}\int\limits_{t/2}^{3t/2}ds\int\limits_{\mathbf{R}^n}|u(x,s)|^p dx.$$

Lemma 2. ([13], *p.2464*). *Let* $p>0$ *and let* $F=(U,V_1,...,V_n)$ *be such that* $V_i\rightrightarrows^x_{y\to\infty} 0$, $i=1,...,n$, $M_p(y,U)\le C$. *Then*

$$M_p(y,\nabla^k F)\le ACy^{-k},\qquad k\in\mathbf{N}.$$

Notation. We denote by $D^k_i f(x,y)$ the partial derivative of the function f of the order k with respect to x_i, $i=1,2,...,n+1$. The notation $f(x,y)\rightrightarrows^x_{y\to\infty} 0$ means that $f(x,y)$ converges to 0 uniformly with respect to x, provided $y\to\infty$, $\nabla^k f(x)=(\frac{\partial^k f(x)}{\partial x_1^k},...,\frac{\partial^k f(x)}{\partial x_n^k})$. Everywhere below the constants $A(k,n),C,K$ depend only on the parameters pointed in parentheses, and may be different from time to time.

3. Proof of Theorem 2.

Lemma 3. *Let* $p>0$, *and let* $F=(U,V_1,...,V_n)$ *satisfy* $M_p(1,F)\le C$. *Then*

$$M_p(y,V_i)\le AC+\int_{\mathbf{R}^n}\Big(\int_y^1\sup_{\xi\ge t}\Big|\nabla U(x,\xi)\Big|dt\Big)^p dx,\qquad i=0,1,...,n,\ V_0=U. \tag{7}$$

Proof. By the Main Theorem of Calculus, and the Cauchy-Riemann equations, we have

$$V_i(x,y)-V_i(x,1)=-\int_y^1\frac{\partial V_i(x,t)}{\partial t}\,dt=-\int_y^1\frac{\partial U(x,t)}{\partial x_i}\,dt, \tag{8}$$

$i=1,2,...,n$,

$$U(x,y)-U(x,1)=-\int_y^1\frac{\partial U(x,t)}{\partial t}\,dt. \tag{9}$$

Then,

$$M_p(y,V_i)\le M_p(1,V_i)+\int_{\mathbf{R}^n}\Big(\int_y^1\sup_{\xi\ge t}\Big|\nabla U(x,\xi)\Big|dt\Big)^p dx,$$

and the result follows. □

Lemma 4. *Let $p > 0$ and let $0 < y < 1$. Then*

$$\int_{\mathbf{R}^n} \Big(\int_y^1 \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt \Big)^p dx \ \leq$$

$$y^p \int_{\mathbf{R}^n} \Big(\sup_{\xi \geq y} |\nabla U(x, \xi)| \Big)^p dx \, + \, 2\, p \int_y^1 t^{p-1} dt \int_{\mathbf{R}^n} \Big(\sup_{\xi \geq t} |\nabla U(x, \xi)| \Big)^p dx. \tag{10}$$

Proof. Denote

$$\Psi(x, y) := \int_y^1 \sup_{\xi \geq t} |\nabla U(x, \xi)| \, dt.$$

Following [6] consider

$$\Phi(x, y) := \Psi(x, y)^p - y^p \Big(- \frac{\partial \Psi(x, y)}{\partial y} \Big)^p, \qquad \Omega(y) := \{ x \in \mathbf{R}^n : \, \Phi(x, y) > 0 \}.$$

By definition of $\Omega(y)$,

$$\int_{\mathbf{R}^n \setminus \Omega(y)} \Psi(x, y)^p dx \leq \, y^p \int_{\mathbf{R}^n \setminus \Omega(y)} \Big(- \frac{\partial \Psi(x, y)}{\partial y} \Big)^p dx. \tag{11}$$

Next, the reasons which are similar to those in [6], imply

$$\int_{\Omega(y)} \Phi(x, y) \, dx - \int_{\Omega(a)} \Phi(x, a) \, dx \, = \int_y^a d\xi \int_{\Omega(\xi)} - \frac{\partial \Phi(x, \xi)}{\partial \xi} \, dx, \ 0 < y < a \leq 1. \tag{12}$$

Moreover, using $\partial^2 \Psi / \partial \xi^2 \geq 0$ for almost every $0 < \xi < 1$, we have

$$- \frac{\partial \Phi}{\partial \xi} = -p \, \Psi^{p-1} \frac{\partial \Psi}{\partial \xi} \, + \, p \, \xi^{p-1} \Big(- \frac{\partial \Psi}{\partial \xi} \Big)^p \, - \, p \, \xi^p \Big(- \frac{\partial \Psi}{\partial \xi} \Big)^{p-1} \frac{\partial^2 \Psi}{\partial \xi^2} \ \leq$$

$$p \Big(- \frac{\partial \Psi}{\partial \xi} \Big) \Big(\Psi^{p-1} \, + \, \xi^{p-1} \Big(- \frac{\partial \Psi}{\partial \xi} \Big)^{p-1} \Big) \ \leq \ 2p \, \xi^{p-1} \Big(- \frac{\partial \Psi}{\partial \xi} \Big)^p.$$

Here the last inequality follows from the definition of $\Omega(\xi)$ and $0 < p < 1$. Since

$$\Psi(x, y) \ \leq \ (1 - y) \, \Big| \frac{\partial \Psi(x, y)}{\partial y} \Big|,$$

the function $\Phi(x, y)$ is negative, provided y is sufficiently close to 1, and we can take a such that $\Omega(a) = \emptyset$. Hence, (12) yields

$$\int_{\Omega(y)} \Phi(x, y) \, dx \ \leq \ 2p \int_y^a \xi^{p-1} \, d\xi \int_{\Omega(\xi)} \Big(- \frac{\partial \Psi(x, \xi)}{\partial \xi} (x, \xi) \Big)^p \, dx \ \leq$$

$$2p \int_y^1 \xi^{p-1} \, d\xi \int_{\mathbf{R}^n} \Big(- \frac{\partial \Psi(x, \xi)}{\partial \xi} (x, \xi) \Big)^p \, dx. \tag{13}$$

Adding

$$\int_{\Omega(y)} \Psi(x, y)^p \, dx \ \leq \ \int_{\Omega(y)} \Phi(x, y) \, dx \, + \, y^p \int_{\Omega(y)} \Big(- \frac{\partial \Psi(x, y)}{\partial y} \Big)^p dx$$

with (11), and using (13), we obtain (10). □

The next result is crucial.

Lemma 5. *Let $p>0$ and let $F=(U,V_1,...,V_n)$ be such that*

$$(14)\qquad 1)\ V_i \rightrightarrows^{x}_{y\to\infty} 0,\ \ i=1,...,n, \qquad 2)\ M_p(y,U)\le C.$$

Then

$$(15)\qquad \Big(\int_{\mathbf{R}^n}\Big(\sup_{\xi\ge y}|\phi_{ij}(x,\xi)|\Big)^p dx\Big)^{1/p}\le ACy^{-1},$$

where $\phi_{ij}(x,y)$ is a coordinate of $\nabla V_i(x,y)$, $j=1,...,n+1$, $x_{n+1}=y$, $i=0,...,n$, $V_0=U$.

Proof. Fix $p>0$ and let $l=\inf\{j\in\mathbf{N}: p>p_j:=(n-1)/(j+n-1)\}$. Since $\nabla V_i(x,y)\rightrightarrows^{x}_{y\to\infty} 0$, we may use the following relation (see [5] or [4])

$$\phi_{ij}(x,y)=\frac{1}{(2l-2)!}\int\limits_y^{\infty}(s-y)^{2l-2}D^{2l-1}_{n+1}\phi_{ij}(x,s)ds=$$

$$\frac{1}{(2l-2)!}\int\limits_0^{\infty}s^{2l-2}D^{2l-1}_{n+1}\phi_{ij}(x,s+y)ds.$$

We have

$$\sup_{\xi\ge y}|\phi_{ij}(x,\xi)|\le R(x,y),$$

where

$$R(x,y):=\frac{1}{(2l-2)!}\int\limits_0^{\infty}s^{2l-2}\Big(\sup_{\xi\ge y}|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+\xi)|\Big)ds.$$

To prove (15) it is enough to show that

$$(16)\qquad M_p(y,R)\le AC\,y^{-1}.$$

Since

$$\Big(|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,\xi)|\Big)^p$$

is subharmonic [3], the function

$$w^p(x,s+y):=\Big(\sup_{\xi\ge y}|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+\xi)|\Big)^p$$

is also subharmonic, and we may apply Theorem 4 (take $a=2l-1$, $A=A(l,n,p)$) to obtain

$$\int\limits_{\mathbf{R}^n}|R(x,y)|^p dx\le A\int\limits_0^{\infty}s^{(2l-1)p-1}ds\int\limits_{\mathbf{R}^n}\Big(\sup_{\xi\ge y}|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+\xi)|\Big)^p dx=$$

$$A\int\limits_0^{\infty}s^{(2l-1)p-1}ds\int\limits_{\mathbf{R}^n}\Big(\sup_{\xi\ge y}|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x,s+\xi)|^{p_j}\Big)^{p/p_j}dx.$$

By the choice of p_j, we have $p/p_j > 1$ and we may use the well-known [4] L^{p/p_j}-boundedness of the maximal operator:

$$\int\limits_{\mathbf{R}^n} |R(x,y)|^p dx \le A \int\limits_0^\infty s^{(2l-1)p-1} ds \int\limits_{\mathbf{R}^n} |\nabla^l D_{n+1}^{l-1} \phi_{ij}(x, s+y)|^p dx.$$

Since $D_{n+1}^{l-1}\phi_{ij}(x,y)$ is the l-th derivative of V_i, we use Lemma 2 to get

$$\int\limits_{\mathbf{R}^n} |\nabla^l D_{n+1}^{l-1} \phi_{ij}(x,y)|^p dx \le \int\limits_{\mathbf{R}^n} |\nabla^{2l} F(x,y)|^p dx \le C\, y^{-2lp}. \tag{17}$$

This gives

$$\int\limits_{\mathbf{R}^n} |R(x,y)|^p dx \le A(l,n,p) C \int\limits_0^\infty s^{(2l-1)p-1} (s+y)^{-2lp} ds = A(l,n,p) C y^{-p},$$

and (16) is proved. □

Proof of Theorem 2. The proof follows from Lemmata 3, 4, 5.

4. Proof of Theorem 3.

Lemma 6. *Let $p > 0$. Then $F = (U, V_1, ..., V_n) \in H^p$ iff*

$$1)\ F \rightrightarrows^{x}_{y\to\infty} 0, \qquad 2) \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \ge y} |U(x,\eta)|\Big)^p dx < C.$$

Proof. Let $F \in H^p$, then both 1) and 2) are well-known, [4]. We prove the converse in two steps. At first we show that

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |U(x,y)|\Big)^p dx \le C. \tag{18}$$

Then we prove that (18) implies

$$(\sup_{y>0} |V_i(\cdot, y)|)^p \in L^1(\mathbf{R}^n), \qquad i = 1, ..., n. \tag{19}$$

To prove (18), we observe that

$$\sup_{y>0} |U(x,y)| = \sup_{y>0} \sup_{\eta \ge y} |U(x,\eta)| = \lim_{y\to 0} \sup_{\eta\ge y} |U(x,\eta)|.$$

Hence, using 2) and Fatou's Lemma, we obtain

$$\int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |U(x,y)|\Big)^p dx \ \le\ \lim_{y\to 0} \int\limits_{\mathbf{R}^n} \Big(\sup_{\eta \ge y} |U(x,\eta)|\Big)^p dx \ \le\ C.$$

It remains to show (19). Using Cauchy-Riemann equations, we have

$$V_i(x,y) = [V_i]_{0\ldots 0}(x,y) = \frac{(-1)^k}{(k-1)!} \int\limits_0^\infty s^{k-1} D_{n+1}^k V_i(x, s+y) ds =$$

$$\frac{(-1)^k}{(k-1)!}\int_0^\infty s^{k-1}D_{n+1}^{k-1}D_iU(x,s+y)ds, \tag{20}$$

where k is chosen such that the function $\Big(\sum\limits_{(j)} U^2_{(j)}(x,y)\Big)^{p/2}$ is subharmonic, ($p\ge p_k=(n-k)/(n+k-1)$, see [3],[4],[16]). Since the expression in (20) is one of the tensor coordinates of $U_{(j)}$, see ([4], page 169), we have

$$\int\limits_{\mathbf{R}^n}\Big(\sup_{y>0}|V_i(x,y)|\Big)^p dx \le \int\limits_{\mathbf{R}^n}\Big(\sup_{y>0}\sum_{(j)} U^2_{(j)}(x,y)\Big)^{p/2}dx =$$

$$\sup_{y>0}\int\limits_{\mathbf{R}^n}\Big(\sum_{(j)} U^2_{(j)}(x,y)\Big)^{p/2}dx \le C\int\limits_{\mathbf{R}^n}\Big(\sup_{y>0}|U(x,y)|\Big)^p dx.$$

The last estimate is proved in [4], page 170. □

Lemma 7. *Let $p>0$ and let $F=(U,V_1,...,V_n)$ be such that $V_i \rightrightarrows^x_{y\to\infty} 0$, $i=1,...,n$. Then $F\in H^p$, provided*

$$1)\ M_p(1,F)\le C, \qquad 2)\ M_p(y,U)\le C, \qquad 3)\ I(p)<\infty. \tag{21}$$

Proof. At first we prove that

$$2^{-p}\int\limits_G\Big(\sup_{y>0}|V_i(x,y)|\Big)^p dx \le AC+\int\limits_G\Big(\sup_{y>0}|\nabla U(x,y)|\Big)^p dx, \qquad \forall G\subset \mathbf{R}^n. \tag{22}$$

This follows from (8), (9) the inequality

$$2^{-p}\Big(\sup_{y>0}|V_i(x,y)|\Big)^p \le \Big(|V_i(x,1)|\Big)^p+\Big(\sup_{y>0}|\nabla U(x,y)|\Big)^p,$$

$j=0,1,2,...,n, V_0=U$, and the first condition in (21).

Now we finish the proof. Assume that the lemma is not true. Then $\forall N>0$ there exists a set $E\subset\mathbf{R}^n$, $0<m(E)<\infty$, such that

$$\int\limits_E\Big(\sup_{y>0}|V_i(x,y)|\Big)^p dx\ge 4N,$$

for some $i=0,1,...,n,\ V_0=U$. Hence, taking $G=E$ in (22) we have

$$\int\limits_E\Big(\sup_{y>0}|\nabla U(x,y)|\Big)^p dx\ge 2N.$$

Then

$$\sup_{y>0}|\nabla U(x,y)| = \sup_{y>0}\sup_{\xi\ge y}|\nabla U(x,\xi)| = \lim_{y\to 0}\sup_{\xi\ge y}|\nabla U(x,\xi)|$$

and Lemma Fatou imply

$$\lim_{y\to 0}\int\limits_E\Big(\sup_{\xi\ge y}|\nabla U(x,\xi)|\Big)^p dx \ge \int\limits_E\Big(\sup_{y>0}|\nabla U(x,y)|\Big)^p dx\ge 2N.$$

But this contradicts the third condition of the lemma, since

$$I(p) \geq \int_0^1 dy \int_{\mathbf{R}^n} \Big(\sup_{\xi \geq y} |\nabla U(x,\xi)| \Big)^p dx \geq \int_0^1 dy \int_E \Big(\sup_{\xi \geq y} |\nabla U(x,\xi)| \Big)^p dx \geq N.$$

□

Lemma 8. *Let $p > 0$ and let $F = (U, V_1, ..., V_n)$ be the harmonic vector satisfying conditions of Theorem 3. Then (18) holds.*

Proof. We argue as in the previous Lemma. We have (9), and

$$2^{-p} \int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} \Big| U(x,y) \Big| \Big)^p dx \leq \int\limits_{\mathbf{R}^n} \Big| U(x,1) \Big|^p dx + \int\limits_{\mathbf{R}^n} \Big(\sup_{y>0} |\nabla U(x,y)| \Big)^p dx$$

leads to a contradiction. □

Proof of Theorem 3. The proof follows from Lemmata 8, 7 and 6.

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