SOME PROBLEMS RELATED TO THE CALDERÓN-ZYGMUND SINGULAR INTEGRAL OPERATORS WITH ROUGH KERNELS.

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Abstract. In this note we discuss several open problems related to rough singular integral operators.

1. Introduction

In this note (see also [9]) we consider singular integral operators which are convolutions on R^n with distributions homogeneous of degree $-n$. There two distinct ways of realizing these operators. The first one is as principal value convolutions (hence the name "singular integrals")

$$Tf(x) \ = a\ f(x) + \lim_{\epsilon\to 0}\int_{|y|\geq\epsilon} f(x-y)\ K(y)\ dy.$$

The second way is as Fourier multipliers,

$$Tf(x) \ = F^{-1}(m(\xi)\widehat{f(\xi)})(x),$$

where the multiplier $m(\xi)$ is homogeneous of degree zero, bounded. The connection between these two representations is given by the identity

$$m(\xi) \ = \ a \ + \int_{\Sigma_{n-1}}(-\frac{\pi i}{2}\ sgn(y\cdot\xi) \ + \ \log\frac{1}{|y\cdot\xi|})\ K(y)\ d\sigma(y),$$

where $d\sigma(y)$ denotes Lebesgue measure on the unit sphere Σ_{n-1}.

We will impose no smoothness conditions on the kernel K (hence the name "rough"), and assume only that $K(y)=|y|^{-n}\Omega(y/|y|)$, where Ω is an integrable function on the unit sphere (or a finite Borel measure), $\Omega\in L^1(\Sigma_{n-1})$, satisfying $\int_{\Sigma_{n-1}}\Omega(\theta)d\theta=0$.

2. Singular integrals generated by spherical measures

The classical result of A. Calderón and A. Zygmund ([2], 1956) says that the operator

$$(T_\Omega f)(x)=\int_{R^n} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}dy \tag{1}$$

is bounded on L^2 provided Ω is **even**, and

$$\operatorname*{essup}_{|\xi|=1}\int_{\Sigma_{n-1}}|\Omega(\theta)|\ \log\frac{1}{|\theta\cdot\xi|}\ d\theta<\infty. \tag{2}$$

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L. Grafakos and A. Stefanov ([8,9]) asked

Question 1. Does condition (2) imply the L^p-boundedness of T_Ω for some $p \neq 2$?

The question is harder than to find out whether a bounded **homogeneous** of degree 0 function is a Fourier multiplier for $p \neq 2$. S. Saeki [17] constructed examples of such functions using the results of deLeeuw [13] and stereographic projection. B. López-Melero [14] constructed an example of a **distribution** Ω from the Sobolev-type spaces, such that T_Ω is bounded for some (but not for all) $p > 1$. L. Grafakos and A. Stefanov [8] proved that

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \, \log^{1+\alpha} \frac{1}{|\theta \cdot \xi|} \, d\theta < \infty, \qquad \alpha > 0, \tag{3}$$

implies the boundedness of T_Ω for $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$.

All results mentioned above indicate that the answer is rather "no", nevertheless, if $n > 2$, Ω is *zonal* (i.e. invariant under all rotations about the x_n-axis) one can obtain a positive answer. Using the methods of J. Duoandikoetxea, J. Rubio de Francia [7] and D. Watson [25], we proved that the corresponding result is true for the operator

$$T_\nu f(x) = \int_0^\infty \frac{dr}{r} \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta), \tag{4}$$

which depends on the finite Borel measure ν (we write $\nu \in M(\Sigma_{n-1})$).

Let $M_z(\Sigma_{n-1})$ be the subspace of $M(\Sigma_{n-1})$, consisting of *zonal* measures.

Theorem ([16]). *Suppose that* $\nu \in M_z(\Sigma_{n-1})$, $\nu(\Sigma_{n-1}) = 0$, $n > 2$. *If*

$$\int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty, \tag{5}$$

then T_ν *extends to a bounded operator from* L^p *into itself for all* $p \in (1, \infty)$.

Corollary ([16]). *Let* $n > 2$, *and let* $\nu \in M_z(\Sigma_{n-1})$ *satisfy*

$$\nu(\Sigma_{n-1}) = 0, \quad \operatorname*{essup}_{|\xi|=1} \int_{\Sigma_{n-1}} \log \frac{1}{|\theta \cdot \xi|} \, d|\nu|(\theta) < \infty. \tag{6}$$

Then T_ν *extends to a bounded operator from* L^p *into itself for all* $p \in (1, \infty)$.

3. Weak-type estimates

Now we turn to questions regarding the behavior of T_Ω on $L^1(R^n)$. In this aspect, there are more question than answers. For example, it is not even known whether $\Omega \in L^\infty(\Sigma_{n-1})$, $f \in L^1(R^n)$ imply the existence of an almost everywhere limit

$$\lim_{\epsilon \to 0} \int_{|y|>\epsilon} f(x-y) \frac{\Omega(y/|y|)}{|y|^n} dy.$$

Definition. T_Ω is said to be of weak type $(1,1)$ if there is a constant $C = C(\Omega) > 0$ such that for all $f \in L^1(R^n)$ we have

$$|\{x : \ |(T_\Omega f)(x)| > \alpha\}| \leq C\|f\|_{L^1}/\alpha.$$

Using the results of M. Christ [4], S. Hofmann [10] proved that the above estimate is true if $\Omega \in L^q(\Sigma_1)$, $q > 1$. Further M. Christ and J.-L. Rubio de Francia [5] proved it for $\Omega \in L\mathrm{Log}^+L(\Sigma_1)$. The latter authors were able to extend their result to all dimensions $n \le 7$ (unpublished). Finally A. Seeger [18] proved that T_Ω is weak type $(1,1)$ bounded when $\Omega \in L\mathrm{Log}^+L(\Sigma_{n-1})$ in all dimensions. After that T. Tao [25] generalized the proof of M. Christ and J.-L. Rubio de Francia for all $n \ge 2$.

Question 2. Let $\Omega \in H^1(\Sigma_{n-1})$ (the Hardy space on the unit sphere [6]). Is T_Ω of weak type $(1,1)$? (The partial answer was obtained by A. Stefanov [20]).

Question 3. Let $0 < \alpha < 1/2$, and let

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \, \frac{1}{|\theta \cdot \xi|^\alpha} \, d\theta < \infty. \tag{7}$$

Is T_Ω of weak type $(1,1)$? (see [19] for recent results in this direction).

Question 4. Let $\nu \in M(\Sigma_{n-1})$ satisfy (5). Is T_ν (see (3)) of weak type $(1,1)$?

Question 5. Let $\nu \in M_z(\Sigma_{n-1})$ satisfy (4). Is T_ν of weak type $(1,1)$?

4. Algebras of singular integral operators

What is the composition of two singular integral operators? This question is of great importance, and is best answered in the Fourier multiplier representation: the composition is an operator of the same form, where the multiplier of the composition is the product of the two multiplies. The problem is that the product of two multipliers answer sheds little light on the relationships between the kernels.

It was shown by Calderón-Zygmund [3] that the composition of two T_{Ω_i} with $\Omega_i \in L^q(\Sigma_{n-1})$, $i = 1, 2$, $q > 1$, is a linear combination of the identity and T_Ω for some $\Omega \in L^q(\Sigma_{n-1})$. On the other hand, it is well-known (see for example [9], [15]), that T_Ω extends to a bounded operator from $L^p(R^n)$ into itself, provided $\Omega \in H^1(\Sigma_{n-1})$. Since $L^q(\Sigma_{n-1}) \subset H^1(\Sigma_{n-1})$, $q > 1$, R. Coifman and G. Weiss [6] asked

Question 6. Do operators T_Ω (together with the identity) form an algebra, when $\Omega \in H^1(\Sigma_{n-1})$?

One can show that (say, if Ω is even),

$$\Omega(\theta) = c_n \, \triangle_{S^{n-1}}^{(n-1)/2} R_{S^{n-1}} m(\theta),$$

where

$$R_{S^{n-1}} m_e(\theta) = \int\limits_{\{\xi \in S^{n-1}:\ \theta \cdot \xi = 0\}} m_e(\xi) d\xi,$$

is the spherical Radon transform, $\triangle$ stands for Laplacian on the sphere. If $n = 2$ the Radon transform disappears, and one can use the technique of Fourier series to prove that $\Omega \in H^1(T)$ if and only if $m' \in H^1(T)$, $\int_T m(e^{i\psi}) d\psi = 0$. This allows to get an affirmative answer [6] in the two-dimensional case, but what if $n \ge 3$?

5. Addendum

It was proved recently (see [8]) that for $0 \le \alpha < 1/2$ one can construct examples of even Ω having mean value zero and satisfying (3) such that the L^2-bounded singular integral operator T_Ω is not bounded on $L^p(\mathbf{R}^d)$ when $\left|\frac{1}{2} - \frac{1}{p}\right| > \alpha$. In particular, one can construct operators T_Ω that are bounded on L^p exactly when $p = 2$.

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