THE FOURIER TRANSFORM AND FIREY PROJECTIONS OF CONVEX BODIES

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Abstract. In this paper we develop a Fourier analytic approach to problems in the Brunn-Minkowski-Firey theory of convex bodies. We study the notion of Firey projections and prove a version of Aleksandrov's projection theorem. We also formulate and solve an analog of the Shephard problem for Firey projections.

1. Introduction

In [F] Firey extended the notion of the Minkowski sum, and introduced, for each real p, a new linear combination of convex bodies, that he called p-sums. Lutwak [Lu2], [Lu3] showed that these Firey sums lead to a Brunn-Minkowski theory for each $p \geq 1$. He introduced the notions of p-mixed volume, p-surface area measure, and proved an integral representation and inequalities for p-mixed volumes, including an analog of the Brunn-Minkowski inequality. As a result, he gave a solution of a generalization of the classical Minkowski problem.

The Fourier analytic approach to sections and projections of convex bodies has recently been developed and has led to several results in the classical Brunn-Minkowski theory. In this paper we apply the Fourier analytic methods to study what we call Firey projections of convex bodies in the context of the Brunn-Minkowski-Firey theory. In particular we consider a generalization of Aleksandrov's projection theorem and formulate and solve an analog of the Shephard problem for Firey projections.

It was proved in [KRZ] that if the surface area measure of a convex body K is absolutely continuous, then

$$\mathrm{Vol}_{n-1}(K|\theta^{\perp}) = -\frac{1}{\pi}\widehat{f_K}(\theta) \qquad \forall\theta \in S^{n-1}, \tag{1}$$

where $K|\theta^{\perp}$ is the orthogonal projection of K onto the hyperplane $\theta^{\perp}$, f_K is the curvature function of the body K extended to a homogeneous of degree $-n-1$ function on $\mathbb{R}^n$, and the Fourier transform is in the sense of distributions. It turns out that this formula can serve as the main tool in the study of different problems concerning the volumes of projections. In particular, it can be applied to the following problem of Shephard.

Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$ and suppose that, for every $\theta \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}(K|\theta^\perp) \leq \mathrm{Vol}_{n-1}(L|\theta^\perp). \tag{2}$$

Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)? \tag{3}$$

This problem was solved independently by Petty [P] and Schneider [Sc1], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. It is also well known [Sc1], that the Shephard problem has an affirmative answer if L is a projection body, i. e. if $\forall \theta \in S^{n-1}$

$$h_L(\theta) = \mathrm{Vol}_{n-1}\left(M|\theta^\perp\right) = \frac{1}{2}\int\limits_{S^{n-1}} |\theta \cdot u| d\, S(M,u)$$

for some convex body M. Here $S(M,\cdot)$ is the surface area measure (see [G]) and $h_L(x) = \max\{x \cdot y : y \in L\}$ is the support function of L. On the other hand the existence of a body which is not a projection body leads to a counterexample. Thus the concept of the projection body represents one of the crucial steps in the solution of the Shephard problem.

The purpose of this paper is to generalize the above facts to the context of the Firey theory. We will use the concept of a p-projection body, introduced by Lutwak [Lu3], [LYZ]. Let $\Pi_p K$, $p \geq 1$ denote the compact convex set whose support function is given by

$$h(\Pi_p K, x)^p = \frac{1}{2n}\int\limits_{S^{n-1}} |x \cdot u|^p d\, S_p(K,u), \qquad x \in \mathbb{R}^n. \tag{4}$$

Here $S_p(K,\cdot)$ is the p-surface area measure; see Section 2 for the definition. A convex body M is called a p-projection body if there is a convex body K such that $M = \Pi_p K$. We say that the support function $h(\Pi_p K,\cdot)$ of $\Pi_p K$ defines Firey projections of a body K.

Our approach is based on the Firey projection analog of (1) (see Section 3):

$$h(\Pi_p K, \xi)^p = \frac{2\pi\, C_p}{n} \widehat{f_p(K,\cdot)}(\xi).$$

Here p is not an even integer, $f_p(K,\cdot)$ is the p-curvature function of the body K, and C_p is a constant depending only on p. We apply this formula together with Lutwak's generalized Minkowski theorem to obtain a generalization of Aleksandrov's projection theorem. The above formula also plays a crucial role in the solution of the following question.

Shephard problem for Firey projections. *Consider two origin-symmetric convex bodies K and L in $\mathbb{R}^n$. Fix $p \geq 1$ and suppose that*

$$\Pi_p K \subseteq \Pi_p L. \tag{5}$$

Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L) \qquad \textit{for } 1 \leq p < n,$$

and

$$\mathrm{Vol}_n(K) \geq \mathrm{Vol}_n(L) \qquad \textit{for } n < p?$$

In the case $p = 1$, condition (5) is equivalent to (2), and so the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. In Section 4, we will show that the answer is negative for any $n \geq 2$ and $p > 1$. Actually, we will prove that the Shephard problem for Firey projections has an affirmative answer if L is a p-projection body, and that the existence of the body which is not a p-projection body leads to a counterexample.

We note that there should be two cases in the Shephard problem for Firey projections. Indeed, consider a convex body K and let $L = rK$ be a dilation of K, $0 < r < 1$. Then (see Section 2)

$$dS_p(rK, \cdot) = h_{rK}^{1-p}(\cdot) dS(rK, \cdot) = r^{n-p} dS_p(K, \cdot).$$

So for $0 < r < 1$ and $n < p$:

$$\Pi_p K \subset \Pi_p(rK),$$

but

$$\mathrm{Vol}_n(K) > \mathrm{Vol}_n(rK).$$

Observe that if $p = n$ then $S_p(rK, \cdot) = S_p(K, \cdot)$, so the n-projection body of K does not contain any information about the size of K.

The paper is organized as follows. In Section 2 we present the necessary auxiliary facts from the Brunn-Minkowski-Firey theory. In Section 3 we prove an analog of formula (1) for $p > 1$, p is not an even integer. Using this formula we prove a generalization of the Aleksandrov's projection theorem. Section 4 is devoted to the solution of then Shephard problem for Firey projections.

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2. The Brunn-Minkowski-Firey Theory

Firey [F] extended the concept of Minkowski sum, and introduced for each real $p \geq 1$, a new linear combination of convex bodies, the so-called p-sums:

$$h(\alpha K +_p \beta L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

Here $K, L \in \mathcal{K}_0$ are origin-symmetric convex bodies, $\alpha, \beta > 0$.

In a series of papers Lutwak [Lu2], [Lu3] showed that the Firey sums lead to a Brunn-Minkowski theory for each $p \geq 1$. He introduced the notion of p-mixed volume, $V_p(K, L)$, $p \geq 1$ as

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon L) - V(K)}{\varepsilon},$$

$K, L \in \mathcal{K}_0$. Lutwak proved that for each $K \in \mathcal{K}_0$, there exists a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} so that

$$V_p(K, L) = \frac{1}{n} \int\limits_{S^{n-1}} h(L, u)^p dS_p(K, u),$$

for all $L \in \mathcal{K}_0$. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, and has the Radon-Nikodym derivative,

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}.$$

If $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, S, we have

$$\frac{dS_p(K, \cdot)}{dS} \equiv f_p(K, \cdot),$$

almost everywhere with respect to S. In this case a convex body $K \in \mathcal{K}_0$ is said to have the p-curvature function $f_p(K, \cdot) : S^{n-1} \to \mathbb{R}$, and hence

$$V_p(K, L) = \frac{1}{n} \int\limits_{S^{n-1}} h(L, u)^p f_p(K, u) dS(u),$$

for all $L \in \mathcal{K}_0$. In particular

$$\mathrm{Vol}(K) = \frac{1}{n} \int\limits_{S^{n-1}} h(K, u)^p f_p(K, u) dS(u). \tag{6}$$

Finally, if a body K has both p-curvature and the curvature functions, then

$$f_p(K, \cdot) = h(K, \cdot)^{1-p} f(K, \cdot).$$

Lutwak [Lu2] generalized the Brunn-Minkowski inequality to the case of p-mixed volumes as follows

$$V_p(K,L)^n \geq \mathrm{Vol}(K)^{n-p}\mathrm{Vol}(L)^p, \qquad p>1.$$

He also proved a generalization of the classical Minkowski theorem, which states that given $p>0$, $p\neq n$, and a continuous even function $g : S^{n-1} \to \mathbb{R}^+$, there exists a unique convex body K such that $f_p(K,\cdot)=g$.

3. Fourier formula for Firey projections

Our main tool is the Fourier transform of distributions. We denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. Every locally integrable real valued function f on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration: for every $\phi \in \mathcal{S}$, $\langle f,\phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$. The Fourier transform of a distribution f is defined by $\langle \hat{f},\hat{\phi}\rangle = (2\pi)^n\langle f,\phi\rangle$, for every test function ϕ.

Let μ be a finite Borel measure on the unit sphere S^{n-1}. We extend μ to a homogeneous distribution of degree $-n-p$. A distribution $\mu_{p,e}$ is called the p-extended measure of μ if, for every even test function $\phi \in S(\mathbb{R}^n)$,

$$\langle \mu_{p,e},\phi\rangle = \int_{S^{n-1}} \langle r_+^{-1-p}, \phi(r\xi)\rangle d\mu(\xi). \tag{7}$$

Here $r_+ = r$ if $r>0$, and $r_+=0$ if $r\leq 0$. In most cases we are only interested in even test functions supported outside of the origin, for which

$$\langle r_+^{-1-p},\phi(r\xi)\rangle = \int_{\mathbb{R}} r_+^{-1-p}\,\phi(r\xi)\,dr = \frac{1}{2}\int_{\mathbb{R}} |r|^{-1-p}\,\phi(r\xi)\,dr;$$

see ([GS], pp. 50, 51) for the general definition of $\langle r_+^{-1-p},\phi(r\xi)\rangle$.

If μ is absolutely continuous with density $g\in L_1(S^{n-1})$, we define the extension $g(x)$, $x\in\mathbb{R}^n\setminus\{0\}$ as a homogeneous function of degree $-n-p$: $g(x)=|x|^{-n-p}\,g(x/|x|)$, and identify $\widehat{\mu_{p,e}}$ with $\widehat{g}$.

Throughout the paper, we write that two homogeneous distributions are equal on the sphere meaning that their homogeneous extensions are equal as distributions on $\mathbb{R}^n$. Recall that the Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.

The following fact is due to Semyanistyi [Se], Formula 4.7. We include here a proof similar to that from [K2], Lemma 3.

Theorem 1. *Let $p > -1$, $p \neq 2k$, $k \in \mathbb{N} \cup \{0\}$. For every $\theta \in S^{n-1}$,*

$$\widehat{\mu_{p,e}}(\theta) = \frac{1}{4\pi\, C_p} \int\limits_{S^{n-1}} |\theta \cdot y|^p d\mu(y),$$

where the constant

$$C_p = \frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)}$$

is positive for each $p \in (4k-2, 4k)$ and negative for $p \in (4k, 4k+2)$.

Proof: Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be an even test function so that $0 \notin \operatorname{supp}(\widehat{\phi})$. Then, by the definition of $\mu_{p,e}$,

$$\langle \widehat{\mu_{p,e}}, \phi \rangle = \langle \mu_{p,e}, \widehat{\phi} \rangle = \frac{1}{2} \int\limits_{S^{n-1}} d\mu(\theta) \int\limits_{-\infty}^{\infty} |r|^{-1-p} \hat{\phi}(r\theta) dr.$$

By Lemma 1 from [K2],

$$\int\limits_{\mathbb{R}^n} |\theta \cdot x|^p \widehat{\psi}(x) dx = (2\pi)^{n-1}\, C_p \int\limits_{\mathbb{R}} |r|^{-1-p} \psi(r\theta) dr$$

for any even $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}(\psi)$. This gives (with $\hat{\psi} = \phi$)

$$\langle \widehat{\mu_{p,e}}, \phi \rangle = \frac{1}{4\pi\, C_p} \int\limits_{S^{n-1}} d\mu(\theta) \int\limits_{R^n} |\theta \cdot \xi|^p \phi(\xi) d\xi =$$

$$\frac{1}{4\pi\, C_p} \int\limits_{R^n} \phi(\xi) d\xi \int\limits_{S^{n-1}} |\theta \cdot \xi|^p d\mu(\theta) =$$

$$\langle \frac{1}{4\pi\, C_p} \int\limits_{S^{n-1}} |\theta \cdot \xi|^p d\mu(\theta), \phi \rangle.$$

Since ϕ is an arbitrary even test function with $0 \notin \operatorname{supp}(\widehat{\phi})$, the distributions $\widehat{\mu_{p,e}}$ and $1/(4\pi\, C_p) \int_{S^{n-1}} |\theta \cdot \xi|^p d\mu(\theta)$ can differ by a polynomial only. But both distributions are even and homogeneous of degree p, so the polynomial must be equal to zero, and $\widehat{\mu_{p,e}}$ coincides with the continuous function $1/(4\pi\, C_p) \int_{S^{n-1}} |\theta \cdot \xi|^p d\mu(\theta)$.

$\square$

The following statement follows from (4) and Theorem 1.

Theorem 2. *Let $p \geq 1$, where p is not an even integer. Then for every $\xi \in S^{n-1}$,*

$$\widehat{S_{p,e}(K,\cdot)}(\xi) = \frac{n}{2\pi\, C_p} h(\Pi_p K, \xi)^p, \tag{8}$$

where C_p is as above. In particular if $S_p(K,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, then

$$\widehat{f_p(K,\cdot)}(\xi) = \frac{n}{2\pi\, C_p} h(\Pi_p K, \xi)^p.$$

Using (8) we may prove the following analog of Aleksandrov's projection theorem:

Theorem 3. *Let $p > 1$, $p \neq n$, where p is not an even integer, and let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$. Then*

$$\Pi_p K = \Pi_p L \qquad \Longleftrightarrow \qquad K = L. \tag{9}$$

Proof: Relation (8) and the uniqueness theorem of the Fourier transform yield $S_{p,e}(K,\cdot) = S_{p,e}(L,\cdot)$. By homogeneity, $S_p(K,\cdot) = S_p(L,\cdot)$ is the same as $S_{p,e}(K,\cdot) = S_{p,e}(L,\cdot)$. It remains to use the uniqueness property of $p-$surface area measures for $p \neq n$ (see [Lu2], Corollary 2.3).

□

Remark. In the case $p = n$, where n is not an even integer, it follows that $\Pi_n K = \Pi_n L$ implies K and L are dilates. The uniqueness of the Fourier transform gives $S_n(K,\cdot) = S_n(L,\cdot)$. But now we may only conclude ([Lu2], Lemma 2.4) that K and L are dilates (because $S_n(K,\cdot)$ does not change under dilation). Theorem 3 is not true for even values of p. Indeed, one can perturb $S_p(K,\cdot)$ (i.e., perturb the body K) without changing $h(\Pi_p K,\xi)$ (see the beginning of the next section for an example of such a perturbation).

Another immediate consequence of Theorem 1 is the characterization of p-projection bodies, where p is not an even integer. The following is equivalent

- L is a p-projection body,
- $\exists\,\mu$ on S^{n-1} such that $C_p \widehat{h_L^p} = \mu_{p,e}$,
- $(\mathbb{R}^n, \|\cdot\|_{L^*})$ is isometric to a subspace of L_p.

The last fact can be found in [K1].

4. Solution of the Shephard problem for Firey projections

Shephard problem for Firey projections. *Consider two origin-symmetric convex bodies K and L in $\mathbb{R}^n$. Fix $p > 1$ and let*

$$\Pi_p K \subseteq \Pi_p L.$$

Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L) \qquad \text{for } 1 \leq p < n,$$

and

$$\mathrm{Vol}_n(K) \geq \mathrm{Vol}_n(L) \qquad \text{for } n < p?$$

We first show that the answer is always negative if p is an even integer. It turns out that for any body $K \in \mathbb{R}^n$ there exists a body L in $\mathbb{R}^n$ such that the Firey projections of bodies K and L are equal but their volumes are different.

Let p be an even integer. Then $|x \cdot \xi|^p = (x \cdot \xi)^p$, and there exists a nonzero continuous even function g on S^{n-1} such that

$$\int_{S^{n-1}} |x \cdot \xi|^p g(x) dx = 0, \qquad \forall \xi \in S^{n-1}. \tag{10}$$

Indeed, if $p = 2k$, then $(x \cdot \xi)^{2k}$ is a polynomial of degree $2k$ with coefficients depending on ξ. So, it is enough to construct a nontrivial even function g, satisfying

$$\int_{S^{n-1}} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} g(x) dx = 0,$$

for all integer powers $0 \leq i_j \leq 2k$ such that $i_1 + i_2 + \dots + i_n = 2k$. Taking $g(x) = \sum_{l=1}^{m} c_l x_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1, \dots, c_m$ provided m is big enough.

Consider an origin-symmetric convex body K in $\mathbb{R}^n$ with a strictly positive p-curvature function (i.e $f_p(K, \xi) > 0$, $\forall \xi \in S^{n-1}$). We may assume that

$$\int_{S^{n-1}} h_K^p(\xi) g(\xi) d\xi \geq 0,$$

(otherwise consider $-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$f_p(K, \xi) - \varepsilon g(\xi) > 0 \qquad \forall \xi \in S^{n-1}.$$

Then we may use the existence theorem for p-curvature functions, (see [Lu2]), to conclude that there exists an origin-symmetric convex body L in R^n such that

$$f_p(L,\xi) = f_p(K,\xi) - \varepsilon g(\xi). \tag{11}$$

Applying (10) we get that $h(\Pi_p L, x) = h(\Pi_p K, x)$, or

$$\Pi_p L = \Pi_p K.$$

But using (6) we have

$$\mathrm{Vol}(K) = \frac{1}{n}\int_{S^{n-1}} h_K^p(x) f_p(K,x)dx = \frac{1}{n}\int_{S^{n-1}} h_K^p(x) f_p(L,x)dx +$$

$$+\frac{\varepsilon}{n}\int\limits_{S^{n-1}} h_K^p(x)g(x)dx \geq V_p(L,K) \geq \mathrm{Vol}(L)^{\frac{n-p}{n}}\mathrm{Vol}(K)^{\frac{p}{n}}.$$

So if $\mathrm{Vol}(K) = \mathrm{Vol}(L)$ then there is an equality in the generalized Brunn-Minkowski inequality and then L and K are dilates (see [Lu2]). Thus $K = L$, but by (11) this contradicts the uniqueness of the p-curvature function.

Theorem 4. *The Shephard problem for Firey projections has a negative answer for any $p > 1$ and $n \geq 2$.*

Proof: Fix $p > 1$, $p \neq n$, p is not an even integer. Then it follows by Theorem 5, below, that for a given dimension n the answer is affirmative if and only if all convex bodies in $\mathbb{R}^n$ are p-projection bodies. This is equivalent to saying that any n-dimensional normed space can be isometrically embedded into L_p, which is not true for any $n \geq 2$ (see [K1]).

□

By an approximation argument (see [Sc2], pp. 158-160), we may assume in the formulation of Shephard's problem that the bodies K and L are such that h_K, h_L are infinitely smooth functions on $\mathbb{R}^n \setminus \{0\}$. Then the Fourier transforms $\widehat{h_K^p}$, $\widehat{h_L^p}$ are the extensions of infinitely differentiable functions on the sphere to homogeneous distributions on $\mathbb{R}^n$ of degree $-n-p$ (see the proof of Lemma 5 from [K3]). Moreover, by the same approximation argument, we may assume that our bodies have absolutely continuous p-surface area measures. Therefore, in the rest of this paper, K and L are origin-symmetric convex bodies with infinitely smooth support functions and absolutely continuous p-surface area measures.

Theorem 5. *Let $p \geq 1$, $p \neq n$, where p is not an even integer.*

(i) If a body L is such that $C_p\widehat{h_L^p}(\theta) \geq 0$ for all $\theta \in S^{n-1}$ then the Shephard problem for Firey projections has an affirmative answer for this L and any K.
(ii) If the curvature function f_K is positive on S^{n-1} and $C_p\widehat{h_K^p}$ is negative on an open subset of S^{n-1} then there exists a body L giving together with K a counterexample in Shephard's problem for Firey projections.

Proof: This theorem will follow from the next two lemmas and the fact that the condition $\Pi_p K \subseteq \Pi_p L$ is equivalent (see Theorem 2) to

$$C_p\widehat{f_p(K,\cdot)}(\theta) \leq C_p\widehat{f_p(L,\cdot)}(\theta), \qquad \theta \in S^{n-1}.$$

Lemma 1. *Consider $p \geq 1$, $p \neq n$, where p is not an even integer. Let K, L be origin-symmetric convex bodies in $\mathbb{R}^n$ and let L be such that h_L is infinitely smooth. Suppose also that the surface area measures of K, L are absolutely continuous. If*

$$0 \leq C_p\,\widehat{h_L^p}(\theta),$$

and

$$C_p\,\widehat{f_p(K,\cdot)}(\theta) \leq C_p\,\widehat{f_p(L,\cdot)}(\theta) \qquad \forall\theta \in S^{n-1},$$

then

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L).$$

Proof: From $C_p\,\widehat{f_p(K,\cdot)}(\theta) \leq C_p\,\widehat{f_p(L,\cdot)}(\theta)$ and $C_p\,\widehat{h_L^p}(\theta) \geq 0$, $\forall\theta \in S^{n-1}$ we get

$$\int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta \leq \int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_p(L,\cdot)}(\theta)d\theta = (*).$$

Using Parseval's formula on the sphere (see Appendix below)

$$(*) = (2\pi)^n \int_{S^{n-1}} h_L^p(\theta)f_p(L,\theta)d\theta = n(2\pi)^n\mathrm{Vol}_n(L).$$

But

$$\int_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta = \int_{S^{n-1}} h_L^p(\theta)f_p(K,\theta)d\theta = n(2\pi)^n V_p(K,L),$$

Thus

$$V_p(K,L) \leq \mathrm{Vol}_n(L).$$

Now we apply the Lutwak's extension of Minkowski inequality:

$$V_p(K,L) \geq \mathrm{Vol}_n(L)^{\frac{p}{n}}\mathrm{Vol}_n(K)^{\frac{n-p}{n}}$$

to get

$$\mathrm{Vol}_n(L) \geq \mathrm{Vol}_n(L)^{\frac{p}{n}}\mathrm{Vol}_n(K)^{\frac{n-p}{n}},$$

or

$$\mathrm{Vol}_n^{n-p}(L) \geq \mathrm{Vol}_n^{n-p}(K).$$

Finally

$$\mathrm{Vol}_n(L) \geq \mathrm{Vol}_n(K), \qquad \text{for } 1 \leq p < n,$$

and

$$\mathrm{Vol}_n(L) \leq \mathrm{Vol}_n(K), \qquad \text{for } n < p.$$

$\square$

Lemma 2. *Consider $p \geq 1$, $p \neq n$, where p is not an even integer, and let K be such that $f_K(\theta) > 0\ \forall \theta \in S^{n-1}$. If $C_p \widehat{h_K^p}$ is negative on an open subset of S^{n-1}, then there exists an origin-symmetric convex body L in $\mathbb{R}^n$, such that*

$$C_p \widehat{f_p(K,\cdot)} \leq C_p \widehat{f_p(L,\cdot)},$$

but

$$\mathrm{Vol}_n(K) > \mathrm{Vol}_n(L).$$

Proof: Let $\Omega = \{\theta \in S^{n-1} : C_p \widehat{h_K^p}(\theta) < 0\}$. Consider a function $v \in C^\infty(S^{n-1})$ such that $C_p\, v$ is a positive even function supported on Ω and v is not identically zero. We extend v to a homogeneous function $r^p v(\theta)$ of degree p on $\mathbb{R}^n$. Then the Fourier transform of $r^p v(\theta)$ is a homogeneous function of degree $-n-p$: $\widehat{r^p v(\theta)} = r^{-n-p} g(\theta)$, where g is an infinitely smooth function on S^{n-1} (see the proof of Lemma 5 from [K3]).

Since g is bounded on S^{n-1} and $f_p(K,\theta) = h_K^{1-p}(\theta) f_K(\theta) > 0$, one can choose a small $\varepsilon > 0$ so that, for every $\theta \in S^{n-1}$ and $r > 0$,

$$f_p(L, r\theta) = f_p(K, r\theta) + \varepsilon r^{-n-p} g(\theta) > 0.$$

By Lutwak's [Lu2] extension of the Minkowski's existence theorem, $f_p(L,\cdot)$ defines an origin-symmetric convex body $L \in \mathbb{R}^n$. By the definition of the function v,

$$C_p \widehat{f_p(L,\cdot)}(r\theta) = C_p \widehat{f_p(K,\cdot)}(r\theta) + \varepsilon\, r^p\, C_p v(\theta) \geq C_p \widehat{f_p(K,\cdot)}(r\theta).$$

Next, since $C_p v$ is supported and is positive in the set where $C_p \widehat{h_K^p} < 0$,

$$\int\limits_{S^{n-1}} \widehat{h_K^p}(\theta) \widehat{f_p(L,\cdot)}(\theta) d\theta = \int\limits_{S^{n-1}} \widehat{h_K^p}(\theta) \widehat{f_p(K,\cdot)}(\theta) d\theta + \int\limits_{S^{n-1}} \widehat{h_K^p}(\theta) \varepsilon v(\theta) d\theta <$$

$$\int\limits_{S^{n-1}} \widehat{h_K^p}(\theta) \widehat{f_p(K,\cdot)}(\theta) d\theta = (*).$$

Now the Parseval's formula (see the appendix below) gives

$$(*) \;=\; (2\pi)^n \int\limits_{S^{n-1}} h_K^p(\theta) f_p(K,\theta)d\theta = n(2\pi)^n \mathrm{Vol}_n(K),$$

and

$$\int\limits_{S^{n-1}} \widehat{h_K^p}(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta = (2\pi)^n \int\limits_{S^{n-1}} h_K^p(\theta) f_p(L,\theta)d\theta =$$

$n(2\pi)^n V_p(L,K)$. Thus $V_p(L,K) < \mathrm{Vol}_n(K)$. As in the previous lemma, this implies

$$\mathrm{Vol}_n(K) > \mathrm{Vol}_n(L), \text{ for } 1 \le p < n,$$

and

$$\mathrm{Vol}_n(K) < \mathrm{Vol}_n(L), \text{ for } n < p.$$

□

Appendix : A version of Parseval's formula on the sphere

$$\int\limits_{S^{n-1}} \widehat{h_L^p}(\theta)\widehat{f_p(K,\cdot)}(\theta)d\theta = (2\pi)^n \int\limits_{S^{n-1}} h_L^p(\theta) f_p(K,\theta)d\theta, \qquad p > 1.$$

We follow [KRZ].

Proposition 1. *Let* $E(t) = t^{2k}e^{-t^4}$, $k > p$, $k \in \mathbb{N}$, $\mu_L(\xi) = \widehat{E(h_L)}(\xi)$. *Then for almost all (with respect to the Lebesgue measure)* $\theta \in S^{n-1}$,

$$\int\limits_0^\infty t^{n-1+p}|\mu_L(t\theta)|dt < \infty.$$

Proof: Since μ_L is a bounded function (h_L is homogeneous of degree 1, so $E(h_L) \in L_1(\mathbb{R}^n)$),

$$\int\limits_0^1 t^{n-1+p}|\mu_L(t\theta)|dt < \infty \qquad \forall \theta \in S^{n-1}.$$

It remains to show that for almost all θ

$$\int\limits_1^\infty t^{n-1+p}|\mu_L(t\theta)|dt < \infty. \tag{12}$$

Let n be even and let m be the first even integer greater than p. One may see that

$$\triangle^{\frac{n+m+2}{2}}\left[h_L^{2k}e^{-h_L^4}\right] \in L_1(\mathbb{R}^n). \tag{13}$$

In fact, after differentiation, the function in front of the exponent is the sum of homogeneous functions of degrees greater than $-n$, and each of them is continuous on the unit sphere.

Now, (13) implies that $\xi \mapsto |\xi|^{n+m+2}\mu_L(\xi),\ \xi \in \mathbb{R}^n$ is a bounded function, since it is the Fourier transform of an L_1-function, hence

$$\int\limits_{|\xi|>1} |\xi|^p|\mu_L(\xi)|d\xi < \int\limits_{|\xi|>1} |\xi|^m|\mu_L(\xi)|d\xi < \infty.$$

Passing to the polar coordinates we get (12).

Finally, one can put $\frac{n+m+1}{2}$ in place of $\frac{n+m+2}{2}$ to prove result for odd n.

□

Proposition 2. *Let μ_L be as above and $\phi \in \mathcal{S}$ with $0 \notin supp(\phi)$. Then*

$$\int\limits_{\mathbb{R}^n}\mu_L(x)dx\int\limits_0^\infty \phi(rx)\frac{dr}{r^{p+1}} = \frac{\Gamma(\frac{2k-p}{4})}{4}\int\limits_{S^{n-1}}\widehat{h_L^p}(\theta)d\theta\int\limits_0^\infty \phi(r\theta)\frac{dr}{r^{p+1}}. \tag{14}$$

Proof: Observe that $\widehat{E(rh_L)}(\xi) = r^{-n}\mu_L(\xi/r)$ for every $r > 0$. We have

$$\int\limits_{\mathbb{R}^n}\mu_L(x)dx\int\limits_0^\infty r^{-p-1}\phi(rx)dr = \int\limits_0^\infty r^{-p-1}dr\int\limits_{\mathbb{R}^n}\phi(rx)\mu_L(x)dx =$$

$$\int\limits_0^\infty r^{-p-1}dr\int\limits_{\mathbb{R}^n}\phi(\xi)\mu_L(\xi/r)r^{-n}d\xi = \int\limits_0^\infty r^{-p-1}dr\int\limits_{\mathbb{R}^n}\widehat{\phi}(y)E(r\,h_L(y))dy =$$

$$= \int\limits_{\mathbb{R}^n}\widehat{\phi}(y)dy\int\limits_0^\infty r^{-p-1}E(r\,h_L(y))dr.$$

By making a substitution $r\,h_L(y) = t$ in the last integral and using the fact that

$$\int\limits_0^\infty t^{2k-p-1}e^{-t^4}dt = \frac{\Gamma((2k-p)/4)}{4},$$

we get the desired result

$$\int_{\mathbb{R}^n} \mu_L(x)dx \int_0^\infty r^{-p-1}\phi(rx)dr = \frac{\Gamma((2k-p)/4)}{4} \int_{\mathbb{R}^n} h_L^p(y)\widehat{\phi}(y)dy =$$

$$= \frac{\Gamma((2k-p)/4)}{4}\langle h_L^p, \widehat{\phi}\rangle = \frac{\Gamma((2k-p)/4)}{4}\langle \widehat{h_L^p}, \phi\rangle =$$

$$= \frac{\Gamma((2k-p)/4)}{4} \int_{\mathbb{R}^n} \widehat{h_L^p}(\xi)\phi(\xi)d\xi = \int_{S^{n-1}} \widehat{h_L^p}(\theta)d\theta \int_0^\infty r^{-p-1}\phi(r\theta)dr.$$

□

Proposition 3. *Let μ_L be as above. Then*

$$\int_0^\infty r^{n-1+p}\mu_L(r\theta)dr = \frac{\Gamma((2k-p)/4)}{4}\widehat{h_L^p}(\theta), \qquad \forall \theta \in S^{n-1}. \tag{15}$$

Proof: Let $\phi \in \mathcal{S}$ be such that $0 \notin \operatorname{supp}(\phi)$. Passing to the polar coordinates, using Proposition 1 and (14), we get

$$\int_{\mathbb{R}^n} \mu_L(x)dx \int_0^\infty \phi(rx)\frac{dr}{r^{p+1}} =$$

$$\int_{S^{n-1}} \left(\int_0^\infty t^{n-1+p}\mu_L(t\theta)dt\right)\left(\int_0^\infty \phi(r\theta)\frac{dr}{r^{p+1}}\right) d\theta =$$

$$= \frac{\Gamma((2k-p)/4)}{4} \int_{S^{n-1}} \widehat{h_L^p}(\theta) \left(\int_0^\infty \phi(r\theta)\frac{dr}{r^{p+1}}\right) d\theta.$$

Now we put $\phi(r\theta) = u(r)v(\theta)$, where v is any infinitely smooth function on S^{n-1} and u is a non-negative test function on $\mathbb{R}$, such that $0 \notin \operatorname{supp} u$. This gives:

$$\int_{S^{n-1}} \Big(\int_0^\infty t^{n-1+p}\mu_L(t\theta)dt\Big)v(\theta)d\theta = \frac{\Gamma((2k-p)/4)}{4} \int_{S^{n-1}} \widehat{h_L^p}(\theta)v(\theta)d\theta$$

for every $v \in C^\infty(S^{n-1})$.

□

Proposition 4. *With the same notation,*

$$\int_{\mathbb{R}^n} f_p(K,x)E(h_L)(x)dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f_p(K,\cdot)}(\xi)\mu_L(\xi)\, d\xi.$$

Proof: First, note that both integrals in the latter formula converge absolutely, because h_L is a homogeneous function of degree 1 and by Proposition 1.

Assume at first that p is not an integer. Since $E(h_L)^{(j)}(0) = 0$, $j = 1, ..., 2k-1$, we have

$$\int_{\mathbb{R}^n} f_p(K,x)E(h_L)(x)dx =$$

$$\int_{S^{n-1}} f_p(K,\theta)\,d\theta \int_0^\infty \frac{E(h_L)(r\theta) - \sum_{j=0}^{[p]} E(h_L)^{(j)}(0)r^j/j!}{r^{p+1}}dr.$$

Let γ_ε be the standard Gaussian density with variance ε. Then the convolution $E(h_L)*\gamma_\varepsilon$ is an *even* test function. Hence, the integral

$$\int_0^\infty \frac{(E(h_L)*\gamma_\varepsilon)(r\theta) - \sum_{j=0}^{[p]}(E(h_L)*\gamma_\varepsilon)^{(j)}(0)r^j/j!}{r^{p+1}}\,dr$$

is well defined for any $\varepsilon > 0$, and is equal to $\langle r_+^{-p-1}, (E(h_L)*\gamma_\varepsilon)(r\theta)\rangle$ (see [GS], p. 51). Splitting it into two integrals over $[0,1]$ and $[1,\infty)$ and using the fact that the derivatives up to order $2k-1$ of the function $F(r) \equiv (E(h_L)*\gamma_\varepsilon)(r\theta)$ are uniformly (with respect to ϵ) bounded, we get

$$\lim_{\epsilon\to 0}\int_0^\infty \frac{(E(h_L)*\gamma_\varepsilon)(r\theta) - \sum_{j=0}^{[p]}\frac{(E(h_L)*\gamma_\varepsilon)^{(j)}(0)r^j}{j!}}{r^{p+1}}\,dr = \int_0^\infty \frac{E(h_L)(r\theta)}{r^{p+1}}\,dr$$

This, together with the definition of $p-$extended measure and the dominated convergence theorem gives:

$$\int_{\mathbb{R}^n} f_p(K,x)E(h_L)(x)\,dx = \lim_{\varepsilon\to 0}\int_{S^{n-1}} \langle r_+^{-p-1}, (E(h_L)*\gamma_\varepsilon)(r\theta)\rangle\, f_p(K,\theta)d\theta =$$

$$\lim_{\varepsilon\to 0}\langle f_p(K,\cdot), E(h_L)*\gamma_\varepsilon\rangle = \lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\langle \widehat{f_p(K,\cdot)}, (E(h_L)*\gamma_\varepsilon)^\wedge\rangle =$$

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f_p(K,\cdot)}(\xi)\,\widehat{E(h_L)}(\xi)\,\widehat{\gamma_\varepsilon}(\xi)\,d\xi =$$

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f_p(K,\cdot)}(\xi)\mu_L(\xi)\,d\xi.$$

In the case p is an integer, the proof follows the same lines. The only difference is that one has to use formulas (5), (6) from ([GS], p. 51) to define $\langle r_+^{-p-1}, (E(h_L) * \gamma_\varepsilon)(r\theta)\rangle$.

$\square$

Proposition 5. *With the same notation,*

$$\frac{\Gamma((2k-p)/4)}{4} \int\limits_{S^{n-1}} f_p(K,\theta) h_L^p(\theta) d\theta =$$

$$\frac{1}{(2\pi)^n} \int_{S^{n-1}} \widehat{f_p(K,\cdot)}(\theta) \int\limits_0^\infty r^{n-1+p} \mu_L(r\theta) dr d\theta.$$

Proof: Note that

$$\frac{\Gamma((2k-p)/4)}{4} h_L^p(\theta) = \int\limits_0^\infty r^{-p-1} E(h_L)(r\theta) dr. \tag{16}$$

Passing to polar coordinates and using (16), Proposition 4, and the fact that $f_p(K,\cdot)$ is a homogeneous function of degree $-n-p$ on $\mathbb{R}^n$, we get

$$\frac{\Gamma((2k-p)/4)}{4} \int\limits_{S^{n-1}} f_p(K,\theta) h_L^p(\theta) d\theta = \int\limits_{\mathbb{R}^n} f_p(K,x) E(h_L)(x) dx =$$

$$\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \widehat{f_p(K,\cdot)}(\xi) \mu_L(\xi) d\xi =$$

$$\frac{1}{(2\pi)^n} \int\limits_{S^{n-1}} \widehat{f_p(K,\cdot)}(\theta) d\theta \int\limits_0^\infty r^{n-1+p} \mu_L(r\theta) dr.$$

$\square$

We finish the proof of Appendix by comparing Propositions 3 and 5.

$\square$

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