# Harmonic Analysis and Convex Geometry, Fall 2013, August 30.

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## Assignment 1.

1. **Problem 1.** Prove the *Radon Theorem*: Each set of affinely dependent point (in particular, each set of at least $n+2$ points) in $\mathbb{R}^n$ can be expressed as the union of two disjoint sets whose convex hulls have a common point.

2. **Problem 2.** For $A \subset \mathbb{R}^n$ the set of all convex combinations of any finitely many elements of $A$ is called the *convex hull* of $A$ and is denoted by conv$A$.

   a) Prove the *Caratheodory Theorem*: If $A \subset \mathbb{R}^n$ and $x \in \text{conv}A$, then $x$ is a convex combination of affinely independent points of $A$. In particular, $x$ is a convex combination of $n+1$ or fewer points in $A$.

   **Hint**: The point $x \in \text{conv}A$ has a representation

   $$x = \sum_{j=1}^{k} \lambda_j x_j, \qquad x_j \in A, \quad \lambda_j > 0, \quad \sum_{j=1}^{k} \lambda_j = 1$$

   with some $k \in \mathbb{N}$. Assume that $k$ is minimal. Using the definition of an affine dependence obtain an affine representation of $x$ with non-negative coefficients at least one of which is zero. Get a contradiction with a minimality of $k$.

   b) Prove that in $\mathbb{R}^n$, the convex hull of a compact set is compact.

   **Hint**: Let $A \subset \mathbb{R}^n$ be a compact set and let $S$ be the set of points

   $$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} : \quad \sum_{j=0}^{n} \alpha_j = 1, \quad \alpha_j \geq 0.$$

   For each point

   $$(\alpha, x) := ((\alpha_0, \alpha_1, \dots, \alpha_n), (x_0, x_1, \dots, x_n)) \in S \times A^{n+1}$$

   let $f(\alpha, x) = \sum\limits_{j=0}^{n} \alpha_j x_j$. Prove that $f(S \times A^{n+1})$ is compact. Conclude that $f(S \times A^{n+1}) = \text{conv}A$.

3. **Problem 3.** Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be convex. If any $n+1$ elements of $\mathcal{M}$ are intersected by some translate of $K$, then all elements of $\mathcal{M}$ are intersected by a translate of $K$.

4. **Problem 4.** Let $A \subset \mathbb{R}^n$ be a set with $\operatorname{diam} A \le 2$. Prove that $A$ lies in a Euclidean ball of radius $\sqrt{\frac{2n}{n+1}}$. If $A$ does not lie in any smaller ball, then the closure of $A$ contains the vertices of a regular $n$-simplex of edge-length 2.

   **Hint**: By the previous problem we can assume that the cardinality of $A$, $\operatorname{card} A \le n+1$. In this case, let $y$ be the center of the smallest Euclidean ball containing $A$ and let $r = r(A)$ be its radius. Denote

$$\{z_0, \dots, z_m\} = \{x \in A : |y - x| = r\}, \qquad m \le n.$$

   Prove that $y \in \operatorname{conv}\{z_0, \dots, z_m\}$. We can assume that $y = 0$. Hence,

$$0 = \sum_{j=0}^{m} \lambda_j z_j, \qquad \sum_{j=0}^{m} \lambda_j = 1, \qquad \lambda_j \ge 0.$$

   Since $4 \ge |z_k - z_j|^2 = 2r^2 - 2z_k \cdot z_j$, show that

$$1 - \lambda_k \ge \sum_{j=0}^{m} \lambda_j \frac{|z_j - z_k|^2}{4} \ge \frac{r^2}{2},$$

   and conclude that $r^2 \le \frac{2m}{m+1}$. What happens if we have $1 - \lambda_k = \frac{r^2}{2}$?