Harmonic Analysis and Convex Geometry, Fall 2013, November 15.

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Assignment 10.

1. **Problem 1.**

   a) Find the curvature of the curve

   $$x = t - \sin t, \quad y = 1 - \cos t, \quad z = 4\sin\frac{t}{2}.$$

   b) Find the curvature of the curve given implicitly by the equations

   $$x + \sinh x = \sin y + y, \quad z + e^z = x + \log(1+x) + 1$$

   at the point $(0,0,0)$.

2. **Problem 2.** Find the expression for the curvature of the plane curve given in polar coordinates.

3. **Problem 3.** Let the convex curve $\boldsymbol{\alpha}$ be a boundary of a convex body $K \subset \mathbb{R}^2$.

   a) Prove that the length of $\boldsymbol{\alpha}$ is given by $L = \int\limits_0^{2\pi} h(\theta)d\theta$, where $h(\theta) := h_K((\cos\theta, \sin\theta))$.

   Hint: Problem 2 b), Assignment 7.

   b) Prove that if K is of constant breadth (width) μ, then $length(\boldsymbol{\alpha}) = \pi\mu$. Conclude that all bodies of the given constant breadth have the same perimeter.

   c) Prove that the line $x\cos\theta + y\sin\theta = h(\theta)$ is tangent to the curve at the point

   $$\boldsymbol{\alpha}(\theta) = (x(\theta), y(\theta)) = (h(\theta)\cos\theta - h'(\theta)\sin\theta, h(\theta)\sin\theta + h'(\theta)\cos\theta).$$

   Hint: A regular convex plane curve can be defined as an *envelope* of its tangent lines. Differentiate $x\cos\theta + y\sin\theta = h(\theta)$ with respect to θ and solve the system.

   (In general, a regular curve $\boldsymbol{\alpha}$ is an envelope of the family of curves $\mathcal{F}$ if at every point $\boldsymbol{\alpha}$ is tangent to at least one of the curves in $\mathcal{F}$ and for every segment of $\boldsymbol{\alpha}$ there are infinitely many points in the segment where $\boldsymbol{\alpha}$ is tangent to curves from $\mathcal{F}$. Use (prove, if you want) the fact that for every point (x,y) of the envelope of the family $\mathcal{F} = \{\gamma_\beta\}_{\beta\in(a,b)}$ of curves (given by the equations $\varphi(x,y,\beta) = 0$) there exists $\beta \in (a,b)$ such that

   $$\varphi(x,y,\beta) = 0, \qquad \frac{\partial}{\partial\beta}\varphi(x,y,\beta) = 0 \;\;).$$

   d) Prove that the radius of curvature of the curve at $\boldsymbol{\alpha}(\theta)$ is $R(\theta) = h(\theta) + h''(\theta)$.

Hint: Differentiate again

$$dx(\theta) = -(h(\theta) + h''(\theta)) \sin\theta d\theta, \quad dy(\theta) = (h(\theta) + h''(\theta)) \cos\theta d\theta.$$

and observe that

$$(x'(\theta), y'(\theta)) \cdot (-\sin\theta, \cos\theta) = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} = \frac{ds}{d\theta}.$$

e) Conclude that the length of $\boldsymbol{\alpha}$ is "an average" of the radius of curvature $L = \int\limits_0^{2\pi} R(\theta)d\theta$.

f) Prove that the area enclosed by $\boldsymbol{\alpha}$ is given by $A = \frac{1}{2}\int\limits_0^{2\pi} h(\theta)R(\theta)d\theta$.

Hint: Use the formula $A = \frac{1}{2}\int\limits_0^{2\pi}(x(\theta)y'(\theta) - x'(\theta)y(\theta))d\theta$ from Calc III.

g) Prove that the area enclosed by $\boldsymbol{\alpha}$ is given by $A = \frac{1}{2}\int\limits_0^{2\pi}\Big(h(\theta)^2 - h'(\theta)^2\Big)d\theta$.

Hint: Integrate by parts.