Harmonic Analysis and Convex Geometry, Fall 2013, November 22.

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Assignment 11.

1. **Problem 1.**

   a) Find the principle curvatures of the paraboloid $z = a(x^2 + y^2)$ at the point $(0, 0, 0)$.

   b) Find the *lines of curvature* of the *Helicoid*

   $$x = u \cos v, \quad y = u \sin v, \quad z = cv.$$

   c) Find the mean and the Gaussian curvature of the paraboloid $z = axy$ at the point $x = y = 0$.

2. **Problem 2.** Let $N(p) = \frac{\boldsymbol{\alpha}_u \times \boldsymbol{\alpha}_v}{|\boldsymbol{\alpha}_u \times \boldsymbol{\alpha}_v|}$ be a Gauss map $N : S \to S^2$ of a regular surface S, associated to a point $p \in S$. Prove *Rodrigues's formula*: $\gamma(t)$ is a line of curvature if and only if

   $$\frac{d}{dt} N(\gamma(t)) = \lambda(t)\gamma'(t)$$

   for some differentiable function $\lambda(t)$. Prove further that the principle curvature in this case is $-\lambda(t)$.

3. **Problem 3.**

   Let α, β, γ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 = 1$, and let K be a convex body in $\mathbb{R}^3$.

   a) Use the fact that $h_K(\lambda(\alpha, \beta, \gamma)) = \lambda h_K((\alpha, \beta, \gamma))$ for any $\lambda > 0$ to obtain

   $$\alpha H_\alpha + \beta H_\beta + \gamma H_\gamma = H,$$

   and

   $$\begin{cases} \alpha H_{\alpha\alpha} + \beta H_{\alpha\beta} + \gamma H_{\alpha\gamma} = 0, \\ \alpha H_{\beta\alpha} + \beta H_{\beta\beta} + \gamma H_{\beta\gamma} = 0, \\ \alpha H_{\gamma\alpha} + \beta H_{\gamma\beta} + \gamma H_{\gamma\gamma} = 0, \end{cases}$$

   where $H = h_K$ and, for example, $H_\alpha = \frac{\partial h_K}{\partial \alpha}$, $H_{\alpha\beta} = \frac{\partial^2 h_K}{\partial \alpha \partial \beta}$. Here K is such that all the derivatives make sense.

   b) Let (x, y, z) be a point of tangency of the plane $\alpha x + \beta y + \gamma z = h_K((\alpha, \beta, \gamma))$ with K, i.e.,

   $$x = H_\alpha, \quad y = H_\beta, \quad x = H_\gamma,$$

and let R be a *principle radius of curvature* of the convex surface F which is a boundary of K near (x, y, z), i.e., the coordinates (ξ, η, ζ) of the corresponding center of curvature are

$$\xi = H_\alpha - R\alpha, \quad \eta = H_\beta - R\beta, \quad \zeta = H_\gamma - R\gamma.$$

Show that the vector $(d\xi, d\eta, d\zeta)$ is parallel to the vector (α, β, γ).

Hint: Move along the corresponding line of curvature of F near $(x, y, z) = \nabla h_K((\alpha, \beta, \gamma))$. Use Problem 2 assuming the normal to F is directed inward.

Conclude that

$$\begin{cases} H_{\alpha\alpha}d\alpha + H_{\alpha\beta}d\beta + H_{\alpha\gamma}d\gamma = Rd\alpha + \lambda\alpha, \\ H_{\beta\alpha}d\alpha + H_{\beta\beta}d\beta + H_{\beta\gamma}d\gamma = Rd\beta + \lambda\beta, \\ H_{\gamma\alpha}d\alpha + H_{\gamma\beta}d\beta + H_{\gamma\gamma}d\gamma = Rd\gamma + \lambda\gamma, \end{cases}$$

for some real number λ.

c) Multiply the above equalities by α, β and γ to observe that $\lambda = 0$. Then prove that R satisfies

$$\begin{vmatrix} H_{\alpha\alpha} - R & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} - R & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} - R \end{vmatrix} = 0.$$

Hint: Use a).

d) Reduce the above cubic (in R) equation to the quadratic one,

$$R^2 - (R_1 + R_2)R + R_1R_2 = 0.$$

Prove that

$$R_1 + R_2 = H_{\alpha\alpha} + H_{\beta\beta} + H_{\gamma\gamma} = \Delta h_K, \qquad R_1R_2 = K_{\alpha\alpha} + K_{\beta\beta} + K_{\gamma\gamma},$$

where the matrix

$$\begin{pmatrix} K_{\alpha\alpha} & K_{\alpha\beta} & K_{\alpha\gamma} \\ K_{\beta\alpha} & K_{\beta\beta} & K_{\beta\gamma} \\ K_{\gamma\alpha} & K_{\gamma\beta} & K_{\gamma\gamma} \end{pmatrix}$$

is the one of the algebraic complements of the matrix of the second derivatives of H,

$$\begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma} \\ H_{\beta\alpha} & H_{\beta\beta} & H_{\beta\gamma} \\ H_{\gamma\alpha} & H_{\gamma\beta} & H_{\gamma\gamma} \end{pmatrix}.$$

Observe that $R_1((\alpha, \beta, \gamma))R_2((\alpha, \beta, \gamma)) = \frac{1}{K(x,y,z)}$ is the reciprocal of the Gaussian curvature K at the point $(x, y, z) = \nabla h_K$.

4. **Problem 4*.** Prove that on the surface of every C^2-regular convex body in $\mathbb{R}^3$ there are at least two umbilical points.