Harmonic Analysis and Convex Geometry, Fall 2013, September 6.

Instructor: Dmitry Ryabogin

Assignment 2.

1. **Problem 1.**

   a) Let W be a convex set. A point $x \in W$ is called *exterior* if it does not lie on any segment with ends in W. Prove that the vertices of the polytope are the only exterior points of it.

   b*) Prove that every non-empty compact convex set has an extreme point.

   Hint: Use induction on dimension.

2. **Problem 2.** Let M be a polytope and let $W \subset V = \operatorname{vert} M$. Then $\operatorname{conv} W$ is a face of M if and only if $\operatorname{aff} W \cap \operatorname{conv}(V \setminus W) = \emptyset$ (here $\operatorname{aff} W$ stands for the *affine hull* of W).

3. **Problem 3.**

   a) Prove that every ridge of a polytope is an intersection of two facets.

   Hint: Let F_k be a facet of an H-polytope M. Then

   $$F_k = H_k \cap \Big(\bigcap_{j \neq k} H_j^+ \Big) = \bigcap_{j \neq k} (H_k \cap H_j^+).$$

   Hence,

   $$R = F_k \cap \operatorname{relbd}(H_k \cap H_j^+) = F_k \cap H_k \cap H_j = K \cap H_k \cap H_j = F_k \cap F_j.$$

   b) Let $F = F^j$, be a *proper* j-face of a d-polytope M, $\dim F^j = j$ and let $j \leq k \leq d-1$. Prove that F is an intersection of at least $k - j + 1$ k-faces containing F.

   Hint: Prove at first that for any F there is a facet containing it: take $x \in \operatorname{rel\,int} F^j$ and observe that x must belong to a supporting hyperplane H generating some facet F^{d-1}. Prove that $F^j \subset H$ and conclude that $F^j \subset F^{d-1}$. Then proceed by induction. Use a) to prove the second part of the theorem: each $(k-1)$-dimensional face of F^{k+1} is an intersection of two k-dimensional faces.

   c) Let F be a face of a d-polytope $M \subset \mathbb{R}^d$, $\dim F = j$. Prove that there exists a $(d-j-1)$-dimensional face F' of M such that $\dim \operatorname{conv}(F \cup F') = d$.

   Hint: In the case $j \leq d-2$ use induction, the fact that each face of a polytope is contained in some facet and a).

4. **Problem 4*.** Let K be a convex body in $\mathbb{R}^3$ such that its orthogonal projections onto all planes (passing through the origin) are polygons. Prove that K is a polytope.