Harmonic Analysis and Convex Geometry, Fall 2013, September 13.

Instructor: Dmitry Ryabogin

Assignment 3.

1. **Problem 1.** Given a polytope M in $\mathbb{R}^n$, let $f_j(M)$ be the numbers of its j-dimensional faces, $0 \le j \le n-1$. Denote by $\mathcal{S}$ a regular simplex, by $B_n^\infty := \mathrm{conv}(\{v_j\}_{j=1}^{2^n})$ a standard cube (here v_j are all possible vectors having ± 1 as its coordinates), and by $B_n^1 := \mathrm{conv}(\pm e_1, \dots, \pm e_n)$ a cross-polytope. Compute $f_j(\mathcal{S})$, $f_j(B_n^\infty)$, and $f_j(B_n^1)$, $0 \le j \le n-1$.

2. **Problem 2.** Prove that any origin-symmetric convex polytope $M \subset \mathbb{R}^n$ with $2f$ facets is a section of a d-dimensional cube for some large d. What is d?

3. **Problem 3.** The $(n-1)$-dimensional *permutahedron* is the convex hull of the $n!$ vectors in $\mathbb{R}^n$ arising by permuting the coordinates of $(1, 2, \dots, n)$.

   a) Verify that it really has $n!$ vertices corresponding to the permutations of $\{1, 2, \dots, n\}$.

   b) Describe all faces of the permutahedron combinatorially (what sets of permutations are vertex sets of faces?).

   c) Determine the dimensions of the faces found in b). In particular, show that the facets correspond to ordered partitions (A, B) of $\{1, 2, \dots, n\}$, $A, B \neq \emptyset$, and count them.

4. **Problem 4*.** Let $\mathcal{S}$ be a (not necessarily regular) simplex in $\mathbb{R}^n$. For every edge e of $\mathcal{S}$ consider the n-dimensional ball B_e having e as its diameter. Show that each $x \in \mathcal{S}$ is covered by at least n balls B_e.

5. **Problem 5*.** Let $\mathcal{C}_1$ be a unit cube in $\mathbb{R}^3$, and let $\mathcal{C}_a$, $\mathcal{C}_b$ be cubes with edges of length a and b. Prove that $a + b \le 1$, provided $\mathcal{C}_a \cup \mathcal{C}_b \subset \mathcal{C}_1$, $int(\mathcal{C}_a) \cap int(\mathcal{C}_b) = \emptyset$. Can you solve a similar problem in $\mathbb{R}^n$?