Harmonic Analysis and Convex Geometry, Fall 2013, September 20.

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Assignment 4.

1. **Problem 1.** Let M and L be two sets in $\mathbb{R}^n$. A *set*

   $$M + L = \{x \in \mathbb{R}^n : x = x_1 + x_2, \quad x_1 \in M, \quad x_2 \in L\}$$

   is a *sum of sets* M and L and is denoted by $M + L$.

   a) Prove that $M + L$ is a polytope, provided M and L are.

   b) Is it true that $2A = A + A$ for any set A? for any *convex* set A?

   c) Is it true that $A + A$ convex implies A is convex?

   d)* Let convex bodies M and L be such that $M + L$ is a polytope. Is it true that M and L are polytopes?

   e)* Let A and B be two convex polytopes. What is the polar of $A + B$?

   f) Let $M = B_3^1 + B_3^\infty$. How many vertices, edges, faces does M have?

2. **Problem 2.** Let $M \subset \mathbb{R}^d$ and $L \subset \mathbb{R}^n$. A *product* of M and L is a set defined as

   $$M \times L = \{(x_1, x_2) : x_1 \in M, \quad x_2 \in L\}.$$

   a) Prove that a product of finitely many polytopes is a polytope.

   b)* Let M and L be two polytopes. What is the polar of $M \times L$?

   c) Let M and L be two polygons in $\mathbb{R}^2$. How many edges does $M \times L \in \mathbb{R}^4$ have?

3. **Problem 3.**

   a) Let $K = [-5, -4]$. Find $(K^*)^*$.

   b) Prove that $(AK)^* = (A^t)^{-1}K^*$ for any linear transformation $A \in GL(n)$ and a convex body K such that $0 \in \text{int}\, K$.

   c) Let K and L be two convex symmetric bodies. Prove that $(L \cap K)^* = \text{conv}\,(K^*, L^*)$.

   d) Prove that the intersection of finitely many polytopes is a polytope.

   e) Prove that the affine image of a polytope is a polytope.

   Hint: $\alpha(M) = \text{conv}\alpha(\text{vert}M)$.

4. **Problem 4*.** Let $K \in \mathbb{R}^3$ be a convex body such that all of its projections onto two-dimensional subspaces (planes passing through the origin) are symmetric. Prove that K is symmetric. Formulate and prove the analogous result in $\mathbb{R}^n$.