Harmonic Analysis and Convex Geometry, Fall 2013, September 27.

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Assignment 5.

1. **Problem 1.**

   a) Prove that any origin-symmetric polytope in $\mathbb{R}^d$ is a projection of a linear image of the octahedron in $\mathbb{R}^f$. What is f?

   c) Prove that for a d-dimensional *simplicial* polytope P we have $\sum_{j=0}^{d} f_j(P) \leq 2^d f_{d-1}(P)$.

2. **Problem 2.** Prove that there are only five regular polytopes in $\mathbb{R}^3$ (a polytope is called *regular* if the amount of edges incident to each vertex is equal to the amount of faces incident to the vertex and all the vertices are equivalent).

3. **Problem 3.**

   a) Prove that any ellipsoid in $\mathbb{R}^3$ centered at the origin has a disc as one of its central sections (by the plane passing through the origin).

   b) Prove that any d-dimensional ellipsoid in $\mathbb{R}^d$ centered at the origin has a k-dimensional ball as one of its central sections (by the k-dimensional subspace). What is k?

4. **Problem 4.** Let $C \subset \mathbb{R}^d$ be a cone starting at the origin, and let S^{d-1} be a unit sphere in $\mathbb{R}^d$, $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. Define a *solid angle* $\sigma(C)$ of C to be a $(d-1)$-volume of the intersection: $\sigma(C) := \text{vol}_{d-1}(C \cap S^{d-1})$. Let v be a vertex of a convex polytope P in $\mathbb{R}^d$. Define the *curvature* ω_v of v to be a solid angle $\sigma(C^*)$ of the *dual cone* C^* of v, i.e., $C^* = \{x \in \mathbb{R}^d : x = \lambda \mathbf{n}_v, \lambda \geq 0\}$, where $\mathbf{n}_v$ are the outer normals to the supporting hyperplanes of P at v.

   Prove the *Gauss-Bonnet* Theorem: the sum of the curvatures of all the vertices of P is the $(d-1)$-volume of S^{d-1}.

5. **Problem 5*.** Let P be a polytope in $\mathbb{R}^d$ containing the origin in its interior, and let F be an r-dimensional face of P, $0 \leq r \leq d-1$. Denote by $V(F)$ an $(r+1)$-dimensional subspace containing F. The polytope P is called r-*equatorial*, if for every F, $P \cap V(F)$ is the union of r-dimensional faces of P (B_1^3 is 1-equatorial, B_∞^3 is not; any P as above is $(d-1)$-equatorial, and any origin-symmetric one is 0-equatorial). Prove that P is r-equatorial if and only if the $(d-r-1)$-dimensional faces of the $(d-r)$-dimensional faces of P^* occur in parallel pairs.