Harmonic Analysis and Convex Geometry, Fall 2013, October 4.

Instructor: Dmitry Ryabogin

Assignment 6.

1. **Problem 1.** Let $\Delta \subset \mathbb{R}^3$ be a tetrahedron. Prove that the following conditions are equivalent:

   (i) all faces of Δ are congruent triangles,

   (ii) all faces of Δ have equal perimeter,

   (iii) all vertices of Δ have equal curvature,

   (iv) the opposite edges of Δ have equal dihedral angles,

   (v) all solid angles of Δ are equal.

   Such Δ are called *equihedral* tetrahedra.

   Hint: (ii) $\Rightarrow$ (i) Write out equations for the edge lengths; (iii) $\Rightarrow$ (i) From the Gauss-Bonnet theorem, conclude that the curvature of each vertex is equal to π. Now write out the equations for the angle sums around each vertex and inside each face; (iv) $\Rightarrow$ (i) Use an argument based on the second proof of the Gauss-Bonnet Theorem given in class; (v) $\Rightarrow$ (iv) Write out equations for the solid angles in terms of dihedral angles.

2. **Problem 2.** Let $P \subset \mathbb{R}^3$ be a convex polytope. Denote by A and B the sums of all solid and dihedral angles, respectively. Prove that $2B - A = 2\pi(|F| - 2)$ where $|F|$ is the number of faces in P.

   Hint: Sum up over vertices.

3. **Problem 3.** Let $P \subset \mathbb{R}^3$ be a convex polytope containing the origin in its interior. For a facet F of P, denote by $\alpha(F)$ the sum of the angles of F and by $\beta(F)$ the sum of the angles of the projection of F onto a unit sphere centered at the origin. Finally, let $\omega(F) = \beta(F) - \alpha(F)$. Prove that $\sum\limits_{F \subset P} \omega(F) = 4\pi$.

4. **Problem 4.** Prove the theorem of Aleksandrov: *Let $P, Q \subset \mathbb{R}^3$ be two combinatorially equivalent convex polytopes with equal corresponding face angles. Then they have equal corresponding dihedral angles.*

5. **Problem 5*.** Two convex polytopes P and Q in $\mathbb{R}^3$ are called *parallel* if they are combinatorially equivalent and the corresponding faces are parallel. Prove the theorem of Aleksandrov: *Assume that the perimeters or areas of parallel faces of parallel polytopes are equal. Then, there exists $a \in \mathbb{R}^3$ such that $P + a = Q$.*