Harmonic Analysis and Convex Geometry, Fall 2013, October 11.

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Assignment 7.

1. **Problem 1.**

   a) Let A be a convex set in $\mathbb{R}^{d-1}$, $d \geq 2$, and let $x = (x_1, \dots, x_{d-1}, x_d) \in \mathbb{R}^d$, $x_d \neq 0$. Prove that $\mathrm{vol}_d(\mathrm{conv}(A, x)) = \frac{1}{d}|x_d|\mathrm{vol}_{d-1}(A)$.

   b) Compute the volume of the Euclidean unit ball $\mathrm{vol}_d(B_2^d)$.

   c) Let ω_{d-1} stand for the $(d-1)$-dimensional surface area measure of the Euclidean unit ball B_2^d, $d \geq 2$. Prove that $\omega_{d-1} = d\mathrm{vol}_d(B_2^d)$.

   d) Prove that for all $n \in S^{d-1}$, $\int\limits_{S^{d-1}} |n \cdot v| d\sigma_{d-1}(v) = 2\mathrm{vol}_{d-1}(B_2^{d-1})$.

   Hint: The left-hand side is precisely twice the $(d-1)$-dimensional volume of the projection of the unit ball.

2. **Problem 2.**

   a) Let P be a convex polytope in $\mathbb{R}^d$ and let $u^\perp$ be a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ orthogonal to a direction $u \in S^{d-1}$. Prove that $2\mathrm{vol}_{d-1}(P|u^\perp) = \sum\limits_{j=1}^{k} |u \cdot n_j| \mathrm{vol}_{d-1}(F_j)$, where F_j, $j = 1, \dots, k$, are the facets of P, and n_j are their normals.

   b) Prove the *Cauchy surface area formula*: $S(P) = \frac{1}{\mathrm{vol}_{d-1}(B_2^{d-1})} \int\limits_{S^{d-1}} \mathrm{vol}_{d-1}(P|u^\perp) d\sigma(u)$, where $S(P)$ is the sum of the $(d-1)$-dimensional volumes of the facets of P.

   Hint: Use d) of the previous exercise.

3. **Problem 3.** A convex polytope Z in $\mathbb{R}^d$ is called a *zonotope* if it is a Minkowski sum of segments, $Z = \sum\limits_{j=1}^{k} [a_j, b_j]$, $a_j, b_j \in \mathbb{R}^d$.

   a) Prove that a zonotope is always centrally-symmetric.

   b) Prove that a translation of a linear image of a zonotope is a projection of the k-dimensional cube.

   c) Let $F(Z, u)$ be a face of a zonotope Z orthogonal to a direction $u \in S^{d-1}$ (F is the intersection of Z and the supporting hyperplane orthogonal to u). Prove that $F(Z, u) = \sum\limits_{j=1}^{k} F([a_j, b_j], u)$. Conclude that any face of a zonotope is itself a zonotope.

   d)* Prove that a polytope in $\mathbb{R}^d$ is a zonotope iff all its two-dimensional faces are symmetric.

   e)* Prove that a polar of a zonotope is not a zonotope.

4. **Problem 4*.** Let P be a convex polytope in $\mathbb{R}^3$ with 7 facets. The area of each facet is 1. Is it possible to estimate the volume of P from below by some fixed positive constant?