Harmonic Analysis and Convex Geometry, Fall 2013, October 25.

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Assignment 8.

1. **Problem 1.**

   A boundary point x of a convex body $K \subset \mathbb{R}^n$ is said to be *singular* if more than one supporting hyperplane of the body goes through it. If only one supporting hyperplane H_x passes through x, the boundary point is called *regular*.

   a) Prove that if a convex body has only regular boundary points, then its boundary can be mapped continuously onto the surface of the n-dimensional unit ball by means of equally directed supporting hyperplanes. If, in addition, $K \cap H_x = \{x\}$ for every boundary point $x \in K$, prove that this map is one to one.

   Let K be a convex body in $\mathbb{R}^n$, $n \geq 2$, and let x be one of its boundary points. Consider the collection of rays emanating from x which contain a point of the body different from x, and adjoint the accumulation points of the point set that arises. The resulting set is a convex cone with vertex x, which is called the *projection cone of the body K at the point x*.

   b) Prove that every supporting hyperplane passing through x is a supporting hyperplane of the cone, and conversely.

   c) Conclude that the projection cone can be obtained as the intersection of all the half-spaces containing the body K whose boundary hyperplanes pass through x.

   d) What is the projection cone at a regular point ?

2. **Problem 2.**

   Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $n \geq 2$, let $F(x) = \|x\|_K$, let x be a boundary point of K, and let y be a variable point.

   a) Prove that the points y that satisfy $F'(x; y - x) \leq 0$ are exactly those that form the projection cone of the body at the point x.

   b) Prove that if x is regular, then $F'(x; y - x)$ must be a linear function of $y - x$.

   c) Conclude that if a convex body has only regular boundary points, then its norm has continuous first partial derivatives.

3. **Problem 3.**

   A *normal cone* (or a *polar cone*) of a convex body $K \subset \mathbb{R}^n$, $n \geq 2$, at the boundary point x is obtained if one erects at x the normals to all the supporting hyperplanes of K at x on the side turned away from the body. If the normal cone at x is $(n-p)$-dimensional, $0 \leq p \leq n$, then x is called a p-face point.

a) Prove that the projection cone at a p-face point x contains a p-dimensional flat (affine subspace) going through x and no flat of higher dimension.

b) Prove that if $F'(x, y)$ has n linearly independent linearity directions, then it is linear.

c) Let F be as in Problem 2 and let x be a p-face point of K. Prove that $F'(x; y)$ has, as a function of y, exactly $p + 1$ linearly independent linearity directions, and conversely.

4. **Problem 4*.** Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $n \geq 3$. Assume also that all sections (by the hyperplanes passing through the origin) of K are absolutely symmetric. Is it true that K is a Euclidean ball?