Harmonic Analysis and Convex Geometry, Fall 2013, November 8.

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Assignment 9.

1. **Problem 1.**

   Given any subset E of $\mathbb{R}^d$ and $\alpha \geq 0$, we define the *exterior α-dimensional Hausdorff measure* of E by $m^*_\alpha(E) = \lim_{\delta \to 0^+} \mathcal{H}^\delta_\alpha(E)$, where

   $$\mathcal{H}^\delta_\alpha(E) := \inf\{ \sum_{k=1}^{\infty} (\, diam\, F_k)^\alpha : \; E \subset \bigcup_{k=1}^{\infty} F_k, \quad diam\, F_k \leq \delta \quad \forall k \},$$

   and $diam\, S = \sup_{x,y \in S} |x - y|$ stands for the diameter of S.

   a) Let $0 < \delta_1 < \delta_2$. Prove that $\mathcal{H}^{\delta_1}_\alpha \geq \mathcal{H}^{\delta_2}_\alpha$. Conclude that the above limit exists (might be infinite).

   b) Prove that $\mathcal{H}^\delta_\alpha(E) \leq m^*_\alpha(E)$.

   c) Prove that $E_1 \subset E_2$ implies $m^*_\alpha(E_1) \leq m^*_\alpha(E_2)$.

   d) Prove that $m^*_\alpha(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} m^*_\alpha(E_j)$ for any countable family of sets in $\mathbb{R}^d$.

   Hint: For each j choose a cover $\{F_{j,k}\}_{k=1}^{\infty}$ of E_j by sets of diameter less than δ such that

   $$\sum_{k=1}^{\infty} (\, diam\, F_{j,k})^\alpha \leq \mathcal{H}^\delta_\alpha(E_j) + \frac{\epsilon}{2^j}.$$

   Then use the fact that $\bigcup_{j,k} F_{j,k}$ is a cover of E.

   e) If the distance $d(E_1, E_2) > 0$, then $m^*_\alpha(E_1 \cup E_2) = m^*_\alpha(E_1) + m^*_\alpha(E_2)$.

   Hint: To show that $m^*_\alpha(E_1 \cup E_2) \geq m^*_\alpha(E_1) + m^*_\alpha(E_2)$ fix $0 < \epsilon < d(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ with sets F_j of diameter less than $0 < \delta < \epsilon$, put $F'_j = E_1 \cap F_j$, $F''_2 = E_2 \cap F_j$ and use

   $$\sum_j (\, diam\, F'_j)^\alpha + \sum_l (\, diam\, F''_l)^\alpha \leq \sum_{k=1}^{\infty} (\, diam\, F_k)^\alpha.$$

   f) Conclude (do not prove, google if necessary) that since m^*_α satisfies all the properties of the metric Caratheodory exterior measure, it is a countably additive measure when restricted to the Borel sets. This measure (we will denote it m_α) is called the *α-dimensional Hausdorff measure.*

g) Prove that the α-dimensional Hausdorff measure is invariant under translations and rotations. Moreover, $\forall \lambda > 0$, $m_\alpha(\lambda E) = \lambda m_\alpha(E)$.

h) Prove that the quantity m_0^* counts the number of points in $E \subset \mathbb{R}$, while $m_1(E) = m(E)$ for all Borel sets $E \subset \mathbb{R}$ (here m denotes the usual Lebesgue measure).

i) Prove that $m_d(E) \leq 2^d c_d m(E)$.

Hint: Let $E \subset \bigcup_j F_j$ be a covering with $\sum_j (\,diam\, F_j)^d \leq m_d(E) + \epsilon$. Find a closed ball B_j centered at a point of F_j so that $B_j \supset F_j$ and $diam\, B_j = 2\, diam F_j$. To get the result, use that

$$m(E) \leq \sum_j m(B_j) = \sum_j c_d(\,diam\, B_j)^d = 2^d c_d \sum_j (\,diam\, F_j)^d \leq 2^d c_d(m_d(E) + \epsilon).$$

2. **Problem 2*.** Let K be a convex body in $\mathbb{R}^d$, $d \geq 2$. The *width* $w_K(\theta)$ of K in the direction $\theta \in S^{d-1}$ is defined as $h_K(\theta) + h_K(-\theta)$. The body K is called *of constant width* if w_K is constant (independent of the direction θ). Prove that in the two-dimensional case among the bodies of the given width the Reuleaux triangle is the one of the smallest area. What is the body of the given constant width of the smallest volume/surface area in $\mathbb{R}^d$, $d \geq 3$?