ON A PAPER OF HUDSON.

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ABSTRACT. We present the detailed proof of Hudson's result in [1].

1. INTRODUCTION

Let $\Omega \in L^1(S^1)$, $f \in L^1(\mathbf{R}^2)$ be nonnegative functions and let $B_r(x)$ be a ball of radius r with the center at x. Consider the maximal operator

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^2} \int\limits_{B_r(x)} f(y)\, \Omega\left(\frac{x-y}{|x-y|}\right) dy.$$

It was proved ([2], [3], [4]) that $\Omega \in L \log L(S^1)$ implies a weak-type (1,1) for M_Ω. It is still an open question whether the $L \log L(S^1)$ condition can be weakened to $L^1(S^1)$. In this note we present the Hudson's result proved in [1], which shows that M_Ω has a weak type (1,1) for some $\Omega \notin L^1 \log L(S^1)$. For example, one can take

$$\Omega(x_1, x_2) = g\left(\frac{x_2}{x_1}\right), \quad g(\theta) = \frac{\chi_{(0,1]}(\theta)}{\theta \log^2(\theta/2)}.$$

More precisely, we have

Theorem (Hudson). *Let $g \in L^1([0,1])$ be monotonically decreasing, such that $\theta g(\theta)$ is monotonically increasing, and let $\Omega(x_1, x_2) = g(x_2/x_1)$. Then*

$$\lambda\, |\{x \in \mathbf{R}^2 : M_\Omega f(x) > \lambda\}| \le c\, (\|g\|_{L^1([0,1])} + 1)\, \|g\|_{L^1([0,1])}\, \|f\|_1. \tag{1}$$

The core of the proof is Lemma 2. It represents an independent interest and might aid the study of the general case $\Omega \in L^1(S^1)$.

All results can be generalized to higher dimensions.

2. SELECTION PROPERTY.

The following definition was introduced by Hudson.

Definition. We say that Ω has the selection property if for any measurable set $D \subset \mathbf{R}^2$ $(0 < |D| < \infty)$ and any positive measurable function $r(x)$ defined on D, there is a measurable subset $E \subseteq D$ such that

$$|E| \ge a|D|, \tag{2}$$

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$$S(E,\Omega,r)(y) \equiv \int\limits_{E\cap B_{r(x)}(y)} \Omega\left(\frac{x-y}{|x-y|}\right)\frac{dx}{r^2(x)} \le A \tag{3}$$

for almost every $y \in \mathbf{R}^2$.

Constants here do not depend on $r(x), D, E$.

Lemma 1. *If Ω has the selection property, then M_Ω is of weak type (1,1).*

Proof. Let $D = \{x \in \mathbf{R}^2 : M_\Omega f(x) > \lambda\}$. We may assume that D is bounded. Then

$$|D| \le \frac{1}{a}|E| \le \frac{1}{a\lambda}\int\limits_E M_\Omega f(x)dx \le \frac{c}{a\lambda}\int\limits_E \frac{dx}{r^2(x)} \int\limits_{B_{r(x)}(x)} f(y)\,\Omega\left(\frac{x-y}{|x-y|}\right)dy =$$

$$\frac{c}{a\lambda}\int\limits_{\mathbf{R}^2} f(y)dy \int\limits_{E\cap B_{r(x)}(y)} \Omega\left(\frac{x-y}{|x-y|}\right)\frac{dx}{r^2(x)} \le c\,\frac{A}{a}\,\frac{\|f\|_1}{\lambda}.$$

□

Thus, to prove the Theorem it is enough to show that Ω satisfies the selection property. We emphersize that the construction of E involves no restriction on $\Omega \in L^1(S^1)$ and $a \simeq 1/\|g\|_{L^1([0,1])}$. All geometrical restrictions come from the estimates of (3) with $A \simeq \|g\|_{L^1([0,1])}$.

3. The Dirac mass estimate

The main idea of the proof is to replace g by Dirac mass δ near the spike. Observe that in this case

$$S(\clubsuit,\delta,r)(y) = \int\limits_{\{x\in\clubsuit:0<x_1-y_1<r(x_1,y_2),x_2=y_2\}} \frac{dx_1}{r(x_1,y_2)}. \tag{4}$$

Claim 1 . *If g is replaced by a Dirac mass supported at 0, then Ω has the selection property. More precisely, for any positive measurable function $r(x)$ defined on a set $D \subset \mathbf{R}^2$ of finite measure there is a set $\widetilde{D} \subseteq D$ satisfying the following properties*

$$|\tilde{D}| > c\,|D|, \qquad S(\widetilde{D},\delta,r)(y) \le C \qquad \textit{for a.e.} \qquad y \in \mathbf{R}^2.$$

This claim follows from the Fubini theorem and the following one-dimensional result.

Lemma 2. *Let $r(t)$ be any positive measurable function on a measurable set $D \subset \mathbf{R}^1(|D| < \infty)$. Then there exists a measurable subset $\widetilde{D} \subset D$ such that*

$$|\tilde{D}| > \frac{1}{20}|D| \tag{5}$$

$$F(y) = \int\limits_{\widetilde{D}\cap\{0<t-y<r(t)\}} \frac{dt}{r(t)} \le C \qquad \textit{for almost all } y \in \mathbf{R}^1. \tag{6}$$

Proof. We will define set $\tilde{D}$ as a union of sets $\{\tilde{D}_i\}_{i=0}^{\infty}$. The procedure described below is a modification of the Calderón-Zygmund stopping time argument. We construct $\tilde{D}_i$ as follows

$$\tilde{D}_i = \{x \in D \mid x \in q \in Q_{i-1} \text{ and } r(x) > |q|\} \cup$$

$$\cup \{x \in D \mid x \notin q \in Q_{i-1} \text{ and } r(x) > 2^{-i}\},$$

where Q_{i-1} is a system of dyadic intervals which we define by induction.

Set $Q_{-1} = \emptyset$ and assume that $Q_0, ..., Q_{i-1}$ have already been constructed. Consider the net of dyadic intervals q with $|q| = 2^{-i}$. The construction of Q_i consists of two steps.

Step 1: We choose from the net all those intervals which do not intersect intervals from Q_{i-1} and for which one of the following conditions holds:

$$(a) \quad |q|^{-1} \int_q F_i(x)dx > 1/2,$$

where

$$F_i(x) = \int_{\tilde{D}_i \cap \{0 < t-x < r(t)\}} \frac{dt}{r(t)},$$

$$(b) \quad \frac{|\tilde{D}_i \cap q|}{|q|} > \frac{1}{2}.$$

Step 2: We add all neighbors from the net to the intervals chosen before ($Q_{i-1} \cup \{$ intervals chosen by step 1 $\}$).

Set $Q_i = Q_{i-1} \cup \{$ intervals chosen by step 1 $\} \cup \{$ intervals chosen by step 2 $\}$. If interval q satisfies (a) and (b) we say that it is chosen by (a).

We claim that $\cup_{i=0}^{\infty} \tilde{D}_i = \tilde{D}$ is the desired set. First of all

$$\tilde{D} \subset D \subset \bigcup_{q \in Q_i, i=0}^{\infty} q. \tag{7}$$

The first inclusion is obvious. The second one follows from the following argument. Fix any $x \in D$. Assume that $r(x) > 2^{-i}$ for some i and x does not belong to any cube from Q_{i-1} (otherwise we are done). Then $x \in \tilde{D}_i$. Since almost all points of $\tilde{D}_i$ are points of density, there is a dyadic interval $q^* \ni x$, $|q^*| = 2^{-j}$, $j \geq i$ and such that $|\tilde{D}_i \cap q^*|/|q^*| > 1/2$. Since $\tilde{D}_i \subseteq \tilde{D}_j$, $|\tilde{D}_j \cap q^*|/|q^*| > 1/2$. Thus by (b), $q^* \subseteq q \in Q_j$.

So (5) will easily follow from (7) and

$$\sum_{q \in Q_i, i=0}^{\infty} |q| \leq 20|\tilde{D}|. \tag{8}$$

To prove (8), let us divide the system $\{Q_i\}_{i=0}^{\infty}$ into three disjoint subsystems: $K_1 = \{$ intervals chosen by condition $(a)\}$, $K_2 = \{$ intervals chosen by condition $(b)\}$, $K_3 = \{$ intervals chosen by step 2 $\}$. Then it is obvious that

$$\sum_{q\in K_3}|q|\le 4\sum_{q\in K_1,q\in K_2}|q|.$$

Moreover

$$\sum_{q\in K_2}|q|=\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_2}|q|\le 2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_2}|\tilde{D}_i\cap q|\le$$

$$2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_2}|\tilde{D}\cap q|\le 2|\tilde{D}|.$$

On the other hand

$$\sum_{q\in K_1}|q|\le\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}|q|\le 2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}\int_q F_i(x)dx\le$$

$$2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}\int_q F(x)dx\le 2\int_R F(x)dx=$$

$$=2\int_R\int_{\tilde{D}\cap\{0<t-x<r(t)\}}\frac{dt}{r(t)}dx=2\int_{\tilde{D}}\frac{1}{r(t)}\int_{\{0<t-x<r(t)\}}dxdt\le 2|\tilde{D}|.$$

So we have

$$\sum_{q\in Q_i,i=0}^{\infty}|q|=\sum_{q\in K_1}|q|+\sum_{q\in K_2}|q|+\sum_{q\in K_3}|q|\le$$

$$\le 5(\sum_{q\in K_1}|q|+\sum_{q\in K_2}|q|)\le 20|\tilde{D}|$$

and it proves (5).

It remains to show (6). At first we observe that $F(y)>1/2$ implies $y\in q\in Q_i$ for some $i\ge 0$. Indeed, fix any y such that $F(y)>1/2$. Since $\tilde{D}_j\subset\tilde{D}_{j+1}$ we have $F_j(y)>1/2$ for sufficiently large j. By the differentiability of integrals, there is a dyadic interval q^*, such that $|q^*|^{-1}\int_{q^*}F_j(x)dx>1/2$ and $|q^*|=2^{-i}$, $i\ge j$. Now $\tilde{D}_j\subset\tilde{D}_i$ implies $|q^*|^{-1}\int_{q^*}F_i(x)dx>1/2$ and (a) gives $q^*\subseteq q\in Q_i$.

Now we decompose $F(y)$ into two parts. The first part will be estimated pointwise, the second one – by mean. Namely

$$F(y)=\int_{L\cap\{0<t-y<r(t)\}}\frac{dt}{r(t)}+\int_{H\cap\{0<t-y<r(t)\}}\frac{dt}{r(t)}=F_L(y)+F_H(y),$$

where, $L=\tilde{D}\cap[y,y+10^2|q|]$ and $H=\tilde{D}\cap[y+10^2|q|,\infty)$. First, let us show that

$$F_L(y)<C. \tag{9}$$

To do so, let us split L into two subsets L_1 and L_2, where $L_1 = \tilde{D} \cap [y + \frac{|q|}{2}, y + 10^2|q|]$ and $L_2 = \tilde{D} \cap [y, y + \frac{|q|}{2}]$. Since $r(t) > t - y \geq \frac{|q|}{2}$ for $t \in L_1 \cap \{t \mid 0 < t - y < r(t)\}$, it is easy to see that

$$F_{L_1}(y) = \int\limits_{L_1 \cap \{0<t-y<r(t)\}} \frac{dt}{r(t)} \leq \frac{2}{|q|} \int\limits_{L_1 \cap \{0<t-y<r(t)\}} dt \leq \frac{2}{|q|} 10^2 |q| \leq C.$$

On the other hand if $t \in L_2$ then $t \in q \in Q_i$ or $t \in \tilde{q} \in Q_{i+1}$ because of the step 2 in construction of Q_{i+1}. In any case, $r(t) > \frac{|q|}{2}$ and

$$F_{L_2}(y) = \int\limits_{L_2 \cap \{0<t-y<r(t)\}} \frac{dt}{r(t)} \leq \frac{2}{|q|} \int\limits_{y}^{y+\frac{|q|}{2}} dt = 1.$$

Combining these two estimates, we get (9).

To finish the proof of the lemma, we should show that

$$F_H(y) \leq C \tag{10}$$

To prove (10), it is enough to show that $F_{H \setminus H_{i-1}}(y) \leq C$ and $F_{H_{i-1}}(y) \leq C$, where $H_{i-1} = \tilde{D}_{i-1} \cap [y + 10^2|q|, \infty)$.

Observe that $F_{H \setminus H_{i-1}}(y) = 0$. Indeed, for any $t \in \tilde{D} \setminus \tilde{D}_{i-1}$, $r(t) \leq 2^{-i+1}$, we have $\tilde{D} \setminus \tilde{D}_{i-1} \cap [y + 10^2|q|, \infty) \cap \{t \mid 0 < t - y < r(t)\} = \emptyset$.

What is left to show is

$$F_{H_{i-1}} \leq C. \tag{11}$$

Let q^* be the closest dyadic interval to q from the right with $|q^*| = 2^{-i+1}$. Since for any $\xi \in q^*$, $H_{i-1} \cap \{t \mid 0 < t - y < r(t)\} \subset H_{i-1} \cap \{t \mid 0 < t - \xi < r(t)\}$, we have

$$F_{H_{i-1}}(y) < \int\limits_{H_{i-1} \cap \{t \mid 0<t-\xi<r(t)\}} \frac{dt}{r(t)} \leq \frac{1}{|q^*|} \int\limits_{q^*} \int\limits_{H_{i-1} \cap \{0<t-\xi<r(t)\}} \frac{dt}{r(t)} d\xi. \tag{12}$$

Observe that

$$q^* \cap p = \emptyset \qquad \text{for any } p \in Q_{i-2} \cup \{\text{intervals choisen by step 1 during stage } i-1\}.$$

Otherwise, by step 2 q would be covered by some interval chosen on stage $l \leq i - 1$. This contradicts the choice of q. Thus

So by (12) and (a) in the construction we have

$$F_{H_{i-1}}(y) \leq \frac{1}{|q^*|} \int\limits_{q^*} \int\limits_{H_{i-1} \cap \{0<t-\xi<r(t)\}} \frac{dt}{r(t)} d\xi \leq \frac{1}{|q^*|} \int\limits_{q^*} F_{i-1}(\xi) d\xi < \frac{1}{2}.$$

□

4. Auxiliary results

The following result is an analogue of the Calderón-Zygmund stopping time procedure. Below $l(q)$ denotes the sidelength of a dyadic cube q.

Claim 2 . *Let $\Omega \in L^1(S^1)$, and let $r(x)$ be any positive measurable function defined on a set $G \subset \mathbf{R}^2$ of finite measure. Then there exist two systems of sets*

$$\{Q_i\}_{i=0}^{\infty} \qquad and \qquad \{E_i\}_{i=0}^{\infty},$$

where Q_i is a system of disjoint dyadic cubes with $l(q) \geq 2^{-i}$, and

$$E_i = \cup_{q \in Q_{i-1}} \{x \in G \cap q : r(x) > l(q)\} \cup \{x \in G \setminus Q_{i-1} : r(x) > 2^{-i}\}$$

such that the following holds:

$$G \subseteq \cup_{i=0}^{\infty} Q_i, \tag{13}$$

$$| \cup_{i=0}^{\infty} E_i| \geq \frac{|G|}{50\,(\,\|\Omega\|_{L^1(S^1)} + 1\,)}, \tag{14}$$

for any ancestor $\tilde{q}$ of $q \in Q_i$, and for any neighbor $\tilde{q}$ of any ancestor of $q \in Q_i$, we have

$$\frac{1}{|\tilde{q}|} \int_{\tilde{q}} S(E_i, \Omega, r)(y)\, dy \; < \; \frac{1}{2}. \tag{15}$$

We set

$$Q \equiv \cup_{i=0}^{\infty} Q_i, \qquad E \equiv \cup_{i=0}^{\infty} E_i \equiv \cup_{q \in Q} \{x \in G \cap q \;:\; r(x) > l(q)\}.$$

Proof of Claim 2. The construction of E_i, Q_i repeats the construction of $\tilde{D}_i, Q_i$ in Lemma 2. One has only to replace the condition (a) in the described stopping time procedure by

$$\frac{1}{|q|} \int\limits_q S(E_i, \Omega, r)(y) dy > \frac{1}{2}.$$

The proof of (13), (14) is similar to the proof of (7), (8). Relation (15) follows by reasons which are similar to those presented at the end of Lemma 2 (see the estimate for $F_{H_{i-1}}$).

□

Claim 3 . *Let $\Omega \in L^1(S^1)$ and let $S(E, \Omega, r)(y) > 1/2$. Then $y \in Q$.*

Proof of Claim 3. Since $E_i \subset E_{i+1}$, we have $S(E_i, \Omega, r)(y) > 1/2$ for sufficiently large i. By the differentiation of integrals $1/|q^*| \int_{q^*} S(E_i, \Omega, r)(y) dy > 1/2$ for some dyadic cube $q^* \ni y$ with $l(q^*) = 2^{-j}, j \geq i$. Now $E_i \subseteq E_j$ implies $1/|q^*| \int_{q^*} S(E_j, \Omega, r)(y) dy > 1/2$ and (a) gives $q^* \subseteq q \in Q_j$.

□

The proof of the following claim is based on the fact that $r(x) \simeq l(q)$ near cube q containing y.

Claim 4 . *Let $\Omega \in L^1(S^1)$ and let $y \in q \in Q_i$ for some $i \geq 0$. Then*

$$S(E \cap B_{10\, l(q)}(y), \Omega, r)(y) \leq C\, \|\Omega\|_{L^1(S^1)}. \tag{16}$$

Proof of Claim 4. Remind that

$$E \equiv \cup_{q\in Q}\{x \in G \cap q \,:\, r(x) > l(q)\},$$

and

$$S(E \cap B_{10\,l(q)}(y), \Omega, r)(y) \equiv \int_{E\cap B_{r(x)}(y)\cap B_{10\,l(q)}(y)} \Omega\left(\frac{x-y}{|x-y|}\right)\,\frac{dx}{r^2(x)}.$$

Now $x \in E \cap B_{r(x)}(y)$ implies $r(x) \geq l(q)/2$. Indeed, this is obvious if $|x-y| \geq l(q)/2$, since $r(x) > |x-y|$. If $|x-y| < l(q)/2$, then either $x \in q$ or x belongs to the neighbor $\tilde{q}$ of q with $l(\tilde{q}) \geq l(q)/2$ (use Step 2). But $x \in E$, so $r(x) \geq l(\tilde{q})$.

This gives

$$S(E \cap B_{10l(q)}(y), \Omega, r)(y) \leq \left(\frac{l(q)}{2}\right)^{-2} \int_{B_{10\,l(q)}(y)} \Omega\left(\frac{x-y}{|x-y|}\right)\,dx \;=\; C\,\|\Omega\|_{L^1(S^1)}.$$

□

5. Estimates on the exceptional set

In this section all proofs strongly depend on the geometric properties of g announced in the Theorem.

The following claim is based on Claim 1 and Claim 2. Let D be any set of finite measure. We apply Claim 1 to get $\tilde{D}$. Now, by Claim 2 we get E from $\tilde{D}$ (take $G = \tilde{D}$).

Claim 5 . *Let $y \in q \in Q_i$ for some i and let*

$$P \;=\; \{x \in E \,:\, y_2 \leq x_2 \leq y_2 + 2\,l(q),\, x_1 \geq y_1 + 2\,l(q)\}. \tag{17}$$

Suppose also that $g \in L^1([0,1])$ and $\theta\, g(\theta)$ is an increasing function. Then

$$S(P, \Omega, r)(y) \leq c\|g\|_{L^1([0,1])}. \tag{18}$$

Proof of Claim 5. Observe that $x \in P \cap B_{r(x)}(y)$ implies

$$2l(q) \leq x_1 - y_1 \leq |x-y| < r(x).$$

This , together with the fact that $\theta\, g(\theta)$ is increasing gives

$$\frac{1}{r(x)} g\left(\frac{x_2-y_2}{x_1-y_1}\right) \leq \frac{1}{2\,l(q)}\, g\left(\frac{x_2-y_2}{2\,l(q)}\right). \tag{19}$$

By (19) we have

$$S(P, \Omega, r)(y) \equiv \int\limits_{P\cap B_{r(x)}(y)} g\left(\frac{x_2-y_2}{x_1-y_1}\right)\frac{dx}{r^2(x)} \leq \frac{1}{2\,l(q)} \int\limits_{P\cap B_{r(x)}(y)} g\left(\frac{x_2-y_2}{2\,l(q)}\right)\frac{dx}{r(x)}.$$

Now we apply the Fubini Theorem to the right-hand side of the above inequality and observe that it does not exceed

$$\frac{1}{2\,l(q)}\int\limits_{y_2}^{y_2+2\,l(q)} g\left(\frac{x_2-y_2}{2\,l(q)}\right)dx_2 \int\limits_{\{x_1>y_1+2l(q):\,x\in P\cap B_{r(x)}(y)\}} \frac{dx_1}{r(x_1,x_2)}.$$

Since $P\subset E\subset \widetilde{D}$, the inner integral is bounded by Claim 1. It remains to make a substitution $x_2-y_2=2\,l(q)\,\theta$ in the outer integral to get

$$S(P,\Omega,r)(y)\le \frac{c}{2\,l(q)}\int\limits_{y_2}^{y_2+2\,l(q)} g\left(\frac{x_2-y_2}{2\,l(q)}\right)dx_2 = c\int\limits_0^1 g(\theta)d\theta.$$

□

The following statement is an analogue of (10).

Claim 6 . *Let $y\in q\in Q_i$ for some i and let $g\in L^1([0,1])$ be decreasing. Then*

(20) $$S(H,\Omega,r)(y)\le C\,\|g\|_{L^1([0,1])},$$

where $H=E\setminus(P\cup B_{10\,l(q)}(y))$.

Proof of Claim 6. Let $H_{i-1}=H\cap E_{i-1}$. We show

$\aleph$: $S(H\setminus H_{i-1},\Omega,r)(y)\le C\,\|g\|_{L^1([0,1])}$.

$\beth$: $S(H_{i-1},\Omega,r)(y)\le C\,\|g\|_{L^1([0,1])}$.

$\aleph$. By the definition of E_{i-1}, we may assume that $r(x)\le 2^{-i+1}=2\,l(q)$. On the other hand, by the reasons which are similar to those in the proof of Claim 4, we have $l(q)\le r(x)$. This gives $\aleph$.

$\beth$. We claim that

(21) $$S(H_{i-1},\Omega,r)(y)\le c\,\frac{1}{|q^*|}\int_{q^*} S(H_{i-1},\Omega,r)(\xi)\,d\xi,$$

where q^* has the same center as q, and $l(q^*)=3l(q)$.

The above relation reads as follows:

$$\int\limits_{H_{i-1}\cap B_{r(x)}(y)} g\left(\frac{x_2-y_2}{x_1-y_1}\right)\frac{dx}{r^2(x)}\le c\,\frac{1}{|q^*|}\int_{q^*}d\xi\int\limits_{H_{i-1}\cap B_{r(x)}(\xi)} g\left(\frac{x_2-\xi_2}{x_1-\xi_1}\right)\frac{dx}{r^2(x)}.$$

By Fubini Theorem,

$$\int\limits_{H_{i-1}\cap B_{r(x)}(y)} g\left(\frac{x_2-y_2}{x_1-y_1}\right)\frac{dx}{r^2(x)}\le c\int\limits_{H_{i-1}}\frac{dx}{r^2(x)}\quad\frac{1}{|q^*|}\int\limits_{q^*\cap B_{r(x)}(x)} g\left(\frac{x_2-\xi_2}{x_1-\xi_1}\right)d\xi.$$

Thus, to show (21) it is enough to prove that for every $x\in H_{i-1}\cap B_{r(x)}(y)$, we have

(22) $$g\left(\frac{x_2-y_2}{x_1-y_1}\right)\le c\,\frac{1}{|q^*|}\int\limits_{q^*\cap B_{r(x)}(x)} g\left(\frac{x_2-\xi_2}{x_1-\xi_1}\right)d\xi.$$

In fact, it is enough to prove the last inequality with K instead of q^*, where $K\subset q^*$: $|K|\ge c\,|q^*|$.

Fix any $x \in H_{i-1} \cap B_{r(x)}(y)$. Define $K(x) \equiv \{\xi \in q^*$, lying above the line $l_{xy}\}$. Since $g(\theta)$ is decreasing observe that

$$g\left(\frac{x_2 - y_2}{x_1 - y_1}\right) \le g\left(\frac{x_2 - \xi_2}{x_1 - \xi_1}\right) \qquad \forall \xi \in K(x).$$

Now let $K \equiv \cap_{x \in H_{i-1} \cap B_{r(x)}(y)} K(x)$. By elementary geometry $|K| \ge c\,|q^*|$. This gives (22). Thus we proved (21).

It remains to show that the mean in the right-hand side of (21) is finite. Let $N(q)$ denote the set of dyadic neighbors of q of the sidelength $l(q)$. Then observe that

$$q^* \subset M \equiv N(\text{father of } q) \cup \text{father of } q.$$

Hence there is a cube $\tilde{q} \in M$ such that

$$\frac{1}{|q^*|} \int\limits_{q^*} S(H_{i-1}, \Omega, r)(\xi)\, d\xi \le c \frac{1}{|\tilde{q}|} \int\limits_{\tilde{q}} S(H_{i-1}, \Omega, r)(\xi)\, d\xi. \tag{23}$$

We claim that the mean in the right-hand side of (23) is bounded by $1/2$. Indeed, observe that

$$\tilde{q} \cap p = \emptyset \qquad \forall p \in Q_{i-2} \cup \{\text{intervals choisen by step 1 during stage}\, i-1\}.$$

Otherwise, by step 2 q would be covered by some cube chosen on stage $l \le i-1$, which contradicts the choice of q. This means that

$$\frac{1}{|\tilde{q}|} \int\limits_{\tilde{q}} S(H_{i-1}, \Omega, r)(\xi)\, d\xi \le \frac{1}{|\tilde{q}|} \int\limits_{\tilde{q}} S(E_{i-1}, \Omega, r)(\xi)\, d\xi < \frac{1}{2}$$

by (a). The proof of $\sqsupset$ is finished.

□

6. Proof of the Theorem

By Claim 3 it is enough to consider $y \in Q_i$ for some $i \ge 0$. We have

$$S(E, \Omega, r)(y) = S(E \cap B_{10\,l(q)}(y), \Omega, r)(y) + S(P, \Omega, r)(y) + S(H, \Omega, r)(y)$$

where P and H are as above. Now claims 4,5, and 6 complete the proof.

□

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