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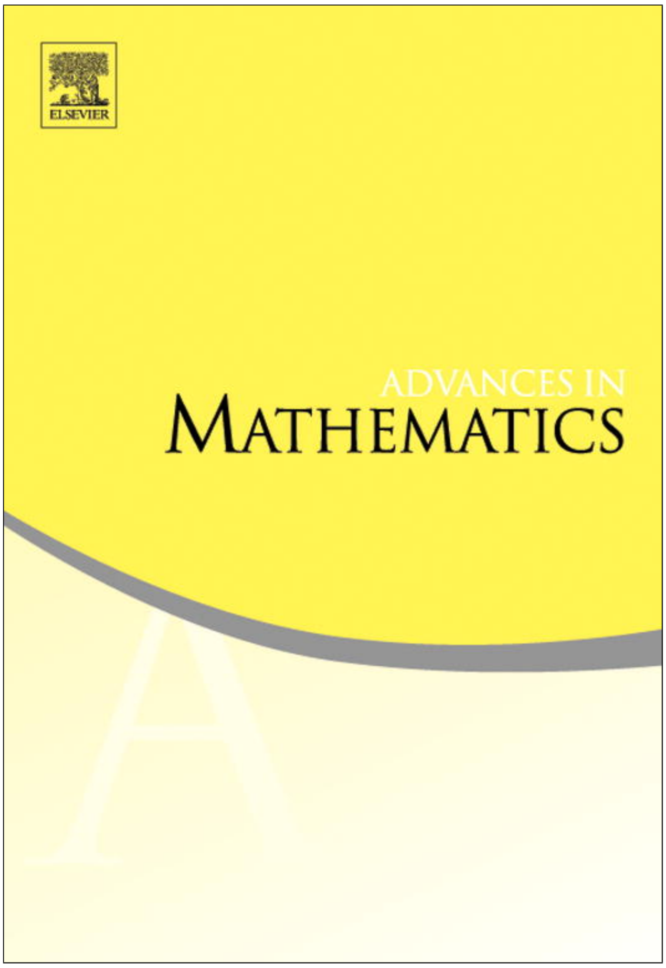

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On the continual Rubik's cube

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Abstract

Let f and g be two continuous functions on the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \geq 3$, and let their restrictions to any one-dimensional great circle E coincide after some rotation ϕ_E of this circle: $f(\phi_E(\theta)) = g(\theta) \forall \theta \in E$. We prove that in this case $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^{n-1}$. This answers the question posed by Richard Gardner and Vladimir Golubyatnikov.
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Keywords: Funk transform; Convex bodies

1. Introduction

The main result of this paper is the following.

Theorem 1. *Let f and g be two continuous functions on the unit sphere S^{n-1} in $\mathbb{R}^n, n \geq 3$, and let their restrictions to any one-dimensional great circle E coincide after some rotation $\phi_E \in SO(2)$ of this circle: $f(\phi_E(\theta)) = g(\theta) \forall \theta \in E$. Then, $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^{n-1}$.*

Theorem 1 gives an answer to the so-called "continual Rubik's cube puzzle", formulated by Richard Gardner and Vladimir Golubyatnikov; see [5, pp. 1,2], and [4].

There are many questions and results about whether the congruency of sections or projections of convex bodies implies the congruency of bodies in the ambient space; see, for example [2, Chapters 3, 7], and [5, Chapters 1–3]. Using Theorem 1 one can easily obtain some results of this type. We have the following.

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Theorem 2. *Let $n \geq 3$ and let K and L be two convex bodies in $\mathbb{R}^n$ containing the origin in their interior. Then $K = L$ or $K = -L$, provided the projections $K|H$, $L|H$ onto any two-dimensional subspace H of $\mathbb{R}^n$ are rotations of each other around the origin.*

Theorem 3. *Let K and L be two star-shaped bodies with respect to the origin in $\mathbb{R}^n$, $n \geq 3$. Then $K = L$ or $K = -L$, provided the sections $K \cap H$, $L \cap H$ by any two-dimensional subspace H of $\mathbb{R}^n$ are rotations of each other around the origin.*

Theorems 2 and 3 shed more light on the subject related to the following open problems (see [2, Problem 3.2, p. 125 and Problem 7.3, p. 289]).

Problem 1. Let $2 \leq k \leq n-1$ and let K and L be two convex bodies in $\mathbb{R}^n$ such that $K|H$ is congruent to $L|H$ for all $H \in \mathcal{G}(n,k)$. Is K a translate of $\pm L$?

Problem 2. Let $2 \leq k \leq n-1$ and let K and L be two star bodies in $\mathbb{R}^n$ such that $K \cap H$ is congruent to $L \cap H$ for all $H \in \mathcal{G}(n,k)$. Is K a translate of $\pm L$?

Here "$K|H$ is congruent to $L|H$" means that there exists an orthogonal transformation $\phi \in O(k)$ such that $\phi(K|H)$ is a translate of $L|H$, $\mathcal{G}(n,k)$ stands for the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$.

If the corresponding projections are translates of each other, or if the bodies are convex and the corresponding sections are translates of each other, the answers to Problems 1 and 2 are known to be affirmative; see [2, Theorems 3.1.3 and 7.1.1]. Thus, one possible way to give the answers to Problems 1 and 2, at least in the case of the direct congruence of the two-dimensional projections (or sections), is to show that there exist the translations of K and L such that the corresponding projections (or sections) of the translated bodies are rotations of each other around the origin.

Theorems 2 and 3 in the convex case were proved by Benjamin Mackey [10] who used the ideas of Vladimir Golubyatnikov, [3,4]. In this case both theorems follow from each other by duality. In connection with Theorem 3, we would also like to mention the result of Rolf Schneider [12], who proved that if K is a convex body in $\mathbb{R}^n$ and p is a point of K such that all intersections of K with hyperplanes through p are congruent, then K is a Euclidean ball.

In this paper we consider only the case $n \geq 3$, the information about the analogues of Theorems 1–3 in the case $n = 2$ is contained in the last section. To prove Theorem 1 we use the techniques from Harmonic Analysis and some simple spherical Topology.

The paper is organized as follows. Since the proof of Theorem 1 is quite long and requires many auxiliary statements, in order for the reader to be able to easily follow the logic of the proof, in the next section we formulate the main auxiliary results, Lemmata 1–3; then we prove Theorems 1–3. In Section 3 we prove Lemmata 3 and 1. In Section 4 we prove several auxiliary results used in the proof of Lemma 2. Lemma 2 is proved in Section 5. (It is similar to Lemma 2.1.4 from [5, p. 17]. We formulate and prove the result for arbitrary positive continuous functions on the unit sphere, Lemma 2.1.4 was formulated in terms of the support functions of convex bodies. Since some details are omitted in [5], we include the proof for the convenience of the reader). In the last section we make some concluding remarks. The proof of technical Lemma 10 is given in the Appendix.

Notation. For $n \geq 2$ we denote by S^{n-1} the unit sphere in $\mathbb{R}^n$, and by $B_t(x)$ the Euclidean n-dimensional ball of radius $t > 0$ centered at $x \in \mathbb{R}^n$. The notation $\xi^{\perp} = \{\theta \in S^{n-1} : \theta \cdot \xi = 0\}$, $\xi \in S^{n-1}$, is used for the great $(n-2)$-dimensional sub-sphere of S^{n-1}. The notation $O(k)$ and $SO(k)$, $2 \leq k \leq n$, for the subgroups of the orthogonal group $O(n)$ and the special

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orthogonal group $SO(n)$ in $\mathbb{R}^n$ is standard. For a two-dimensional subspace E of $\mathbb{R}^n$ we will write $\phi_E \in SO(2)$ meaning that there exists a proper choice of an orthonormal basis in $\mathbb{R}^n$ and a rotation $\Phi \in SO(n)$, with a matrix written in this basis, such that the action of Φ on E is the rotation ϕ_E in E, and the action of Φ on $E^{\perp}$ is trivial, i.e., $\Phi(y) = y \forall y \in E^{\perp}$; here $E^{\perp}$ stands for the orthogonal complement of E. We set

$$\Xi_0 = \{\xi \in S^{n-1} : f(\theta) = g(\theta)\ \forall \theta \in \xi^{\perp}\}, \tag{1}$$

$$\Xi_\pi = \{\xi \in S^{n-1} : f(\theta) = g(-\theta)\ \forall \theta \in \xi^{\perp}\}, \tag{2}$$

where f and g are any functions on S^{n-1}. Given a function f on S^{n-1} we let

$$f_e(\theta) = \frac{f(\theta) + f(-\theta)}{2}, \quad \forall \theta \in S^{n-1},$$

stand for its even part.

2. Proofs of Theorems 1–3

The following results will be used in the proof of Theorem 1. Their proofs are given in the subsequent sections.

Lemma 1. *Let $n \geq 3$ and let $S^{n-1} = \Xi_0 \cup \Xi_\pi$, where f and g are continuous. Then $f(\theta) = g(\theta) \forall \theta \in S^{n-1}$ or $f(\theta) = g(-\theta) \forall \theta \in S^{n-1}$.*

Lemma 2. *Let $n = 3$ and let f and g be two positive continuous functions satisfying the conditions of Theorem 1. Then,*

$$S^2 = \Xi_0 \cup \Xi_\pi \cup \Sigma, \tag{3}$$

where Σ is the set of all directions $\xi \in S^2$ such that

$$f_e(\theta) = g_e(\theta) = \text{const}, \quad \forall \theta \in \xi^{\perp}. \tag{4}$$

Observe that the constant is independent of $\xi \in \Sigma$, since any two great sub-spheres of S^2 intersect.

Lemma 3. *Let $n = 3$ and let f, g and Σ be as in Lemma 2. Then,*

$$\Sigma \subseteq (\Xi_0 \cup \Xi_\pi). \tag{5}$$

The idea of the proof of Theorem 1 is that the restrictions of f and g onto big circles "do not rotate". If they do, due to the fact that f^2, g^2 satisfy the conditions of Theorem 1 as long as f and g do, using Lemma 2, one can reduce everything to two equations with two unknown variables, cf. (6).

Proof of Theorem 1. Let $n = 3$. We observe that by adding a constant we can assume that f and g are both positive. By Lemmata 2 and 3 we have $S^2 = \Xi_0 \cup \Xi_\pi$. Hence, the result follows from Lemma 1.

Let $n = 4$, and let E be *any* two-dimensional sub-sphere of S^3. Consider all one-dimensional sub-circles of E. Since $E \subset S^3$, they are also one-dimensional sub-circles of S^3, hence, we see that the conditions of Theorem 1 are satisfied for $E = S^2$ and $n = 3$. Applying our result in the case $n = 3$, we conclude that $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta) \forall \theta \in E$. Since the chosen E was arbitrary, we can apply Lemma 1 for $n = 4$ to see that Theorem 1 holds in this case.

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Finally, assume by induction that for all $\xi \in S^{n-1}$, the result holds for the restrictions $f|_{\xi^\perp}$, $g|_{\xi^\perp}$ of f and g onto *any* $(n-2)$-dimensional sub-spheres $\xi^\perp$ of S^{n-1}, i.e.,

$$f|_{\xi^\perp}(\theta) = g|_{\xi^\perp}(\theta) \quad \forall\theta \in \xi^\perp, \quad \text{or} \quad f|_{\xi^\perp}(\theta) = g|_{\xi^\perp}(-\theta) \quad \forall\theta \in \xi^\perp.$$

Then, we again apply Lemma 1, and the result follows. □

Proof of Theorem 2. Let $x \in \mathbb{R}^n$ and let $h_K(x) = \max\{x \cdot y : \ y \in K\}$ be the *support function* of the compact convex set $K \subset \mathbb{R}^n$, (see [2, p. 16]).

We are given that for every two-dimensional subspace H there exists $\psi = \psi_H \in SO(2)$ such that the projections of the bodies K and L onto H satisfy $\psi(K|H) = L|H$. Hence, $h_{\psi(K|H)}(x) = h_{L|H}(x)$ for all $x \in H$. Since the support function is homogeneous of degree 1, we have $h_{\psi(K|H)}(\theta) = h_{L|H}(\theta)$ for all $\theta \in H \cap S^{n-1}$. By the well-known properties of the support function,

$$h_{K|H}(\theta) = h_K(\theta), \qquad h_{\psi(K|H)}(\theta) = h_{K|H}(\psi^t(\theta)), \quad \forall\theta \in H \cap S^{n-1},$$

(see, for example, (0.21), (0.26), [2, pp. 17,18]), we obtain

$$h_K(\phi(\theta)) = h_L(\theta) \quad \forall\theta \in H \cap S^{n-1},$$

where $\phi = \psi^t$. It remains to apply Theorem 1 with $f = h_K, g = h_L$ to conclude that $h_K(\theta) = h_L(\theta)$ or $h_K(\theta) = h_L(-\theta)$ for all $\theta \in S^{n-1}$. In the first case, $K = L$, and in the second, $K = -L$. □

Proof of Theorem 3. Let $x \in \mathbb{R}^n \setminus \{0\}$, let $K \subset \mathbb{R}^n$ be a star-shaped set, and let $\rho_K(x) = \max\{c : \ cx \in K\}$ be its *radial function*, where the line through x and the origin is assumed to meet K, (see [2, p. 18]).

We are given that for every two-dimensional subspace H there exists $\psi = \psi_H \in SO(2)$ such that the sections of the bodies K and L satisfy $\psi(K \cap H) = L \cap H$. Hence, $\rho_{\psi(K\cap H)}(x) = \rho_{L\cap H}(x)$ for all $x \in H$. Since the radial function is homogeneous of degree -1, we have $\rho_{\psi(K\cap H)}(\theta) = \rho_{L\cap H}(\theta)$ for all $\theta \in H \cap S^{n-1}$. By the well-known properties of the radial function,

$$\rho_{K\cap H}(\theta) = \rho_K(\theta), \qquad \rho_{\psi(K\cap H)}(\theta) = \rho_{K\cap H}(\psi^{-1}(\theta)), \quad \forall\theta \in H \cap S^{n-1},$$

(see, for example, (0.33), [2, p. 20]), we obtain

$$\rho_K(\phi(\theta)) = \rho_L(\theta) \quad \forall\theta \in H \cap S^{n-1},$$

where $\phi = \psi^{-1}$. It remains to apply Theorem 1 with $f = \rho_K, g = \rho_L$ to conclude that $\rho_K(\theta) = \rho_L(\theta)$ or $\rho_K(\theta) = \rho_L(-\theta)$ for all $\theta \in S^{n-1}$. In the first case, $K = L$, and in the second, $K = -L$. □

3. Proofs of Lemmata 3 and 1

To prove Lemma 3 we will need the following well-known result. We will apply it in a way that is very similar to the one in [12].

Lemma 4. *Let $n = 3$ and let f be a positive continuous function on S^2. If there exist two constants c_1, c_2 such that for some $\xi \in S^2$,*

$$f(\theta) + f(-\theta) = 2c_1 \quad \textit{and} \quad f^2(\theta) + f^2(-\theta) = 2c_2 \quad \forall\theta \in \xi^\perp, \tag{6}$$

then $f(\theta) = c_1$ for every $\theta \in \xi^\perp$.

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Proof. Since $f(\theta) + f(-\theta) = 2c_1$ for all $\theta \in \xi^{\perp}$, there exists $\theta_0 \in \xi^{\perp}$ such that $f(\theta_0) = f(-\theta_0)$.

Indeed, we can assume that f is not identically constant and consider the function $g(\theta) := f(-\theta) - f(\theta) = 2c_1 - 2f(\theta)$ for all $\theta \in \xi^{\perp}$. By the intermediate value theorem, it is enough to show that there exist $\theta_1, \theta_2 \in \xi^{\perp}$ such that $g(\theta_1) > 0$ and $g(\theta_2) < 0$, (or $g(\theta_1) < 0$ and $g(\theta_2) > 0$). If $g(\theta) > 0$ for all $\theta \in \xi^{\perp}$, then $f(\theta) < c_1$ for all $\theta \in \xi^{\perp}$. But then, $f(\theta) + f(-\theta) < 2c_1$, a contradiction. Similarly, if $g(\theta) < 0 \forall \theta \in \xi^{\perp}$, then $f(\theta) + f(-\theta) > 2c_1$, a contradiction. Thus, g must change the sign, and $\exists \theta_0$ such that $g(\theta_0) = 0$.

Now we take this θ_0 and substitute it into the first relation in (6) to obtain $c_1 = f(\theta_0)$. Using the second relation in (6), we also see that $c_2 = f^2(\theta_0) = c_1^2$.

Fix any $\theta \in \xi^{\perp}$. Then, (6) can be rewritten as

$$x + y = 2c_1, \qquad x^2 + y^2 = 2c_1^2,$$

where $x = f(\theta) > 0$, and $y = f(-\theta) > 0$. Since the system has a unique solution $x = y = c_1$ and $\theta \in \xi^{\perp}$ was arbitrary, the result follows. $\square$

Proof of Lemma 3. We observe that if two positive functions f and g satisfy the conditions of Theorem 1, then f^2 and g^2 satisfy the same conditions as well. Hence, we may apply Lemma 2 to f^2 and g^2 instead of f and g, and we can assume that the corresponding sets $\Xi_0, \Xi_\pi, \Sigma \setminus (\Xi_0 \cup \Xi_\pi)$, are the same for f, g and f^2, g^2. In fact, for positive functions $f = g$ is equivalent to $f^2 = g^2$, and it is clear that the corresponding sets Ξ_0, Ξ_π, defined for f, g and f^2, g^2, coincide. Since, by Lemma 2, we have $S^2 \setminus (\Xi_0 \cup \Xi_\pi) = \Sigma \setminus (\Xi_0 \cup \Xi_\pi)$, we see that the corresponding sets $\Sigma \setminus (\Xi_0 \cup \Xi_\pi)$ coincide as well.

We claim that $\Sigma \setminus (\Xi_0 \cup \Xi_\pi) = \varnothing$. Indeed, if $\Sigma \setminus (\Xi_0 \cup \Xi_\pi)$ were not empty, then for any $\xi \in (\Sigma \setminus (\Xi_0 \cup \Xi_\pi))$, we would have (4) and the analogue of (4) for f^2, g^2 instead of f, g. In other words, we would have (6) and the analogue of (6) for g instead of f. Then, applying Lemma 4, we would obtain $f(\theta) = c_1$ (and $g(\theta) = c_1$) for all $\theta \in \xi^{\perp}$. Hence, f and g are constant functions on $\xi^{\perp}$, and we would get $\xi \in (\Xi_0 \cup \Xi_\pi)$, a contradiction. Thus, $\Sigma \setminus (\Xi_0 \cup \Xi_\pi) = \varnothing$ and (5) follows from (3). $\square$

To prove Lemma 1 we will use the following.

Lemma 5. *Let $n \geq 3$ and let f and g be two continuous functions on S^{n-1}. Then, the sets Ξ_0 and Ξ_π are closed.*

Proof. We prove that Ξ_0 is closed. The proof for Ξ_π is similar.

We can assume that Ξ_0 is non-empty. Let $(\xi_m)_{m=1}^{\infty}$ be a sequence of elements of Ξ_0 converging to $\xi \in S^{n-1}$, and let θ be any point on $\xi^{\perp}$.

It is readily seen that there exists a sequence $(\theta_l)_{l=1}^{\infty}$, $\theta_l \in \xi_l^{\perp}$, converging to θ as $l \to \infty$. Indeed, let $B_{\frac{1}{l}}(\theta)$ be a Euclidean ball centered at θ of radius $\frac{1}{l}$, where $l \in \mathbb{N}$. Since $\xi_m^{\perp} \to \xi^{\perp}$ as $m \to \infty$, for each $l \in \mathbb{N}$ there exists $m = m(l)$ such that

$$\xi_m^{\perp} \cap (B_{\frac{1}{l}}(\theta) \cap S^{n-1}) \neq \varnothing.$$

Choose any $\theta_l = \theta_{m(l)} \in \xi_{m(l)}^{\perp} \cap B_{\frac{1}{l}}(\theta)$. Then $\theta_l \to \theta$ as $l \to \infty$.

Finally, if $(\theta_l)_{l=1}^{\infty}$, $\theta_l \in \xi_l^{\perp}$, is any sequence of points converging to θ as $l \to \infty$, then $f(\theta_l) = g(\theta_l)$ for all $l = 1, 2, \ldots$, yields $f(\theta) = g(\theta)$. Hence, $\xi \in \Xi_0$. $\square$

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Before we start proving Lemma 1, we observe that for any $\xi_0 \in S^{n-1}$, we have

$$S^{n-1} = \bigcup_{\xi \in \xi_0^{\perp}} \xi^{\perp}. \tag{7}$$

Indeed, in the case ξ_0 being the north pole, $\xi_0 = (0, \dots, 0, 1)$, (7) can be checked directly, using the definition of the inner product in $\mathbb{R}^n$. In the general case, (7) is a consequence of the result for the north pole and the transitivity of the action of the group of rotation on the manifold $\{\xi^{\perp}\}$ of all great sub-spheres of S^{n-1}.

Proof of Lemma 1. We can assume that the sets Ξ_0, Ξ_π are not empty. We can also assume that $\Xi_0 \cap \Xi_\pi \neq \varnothing$. Indeed, let ξ be a point on the boundary of Ξ_0, ($\xi \in \Xi_0$, since Ξ_0 is closed). Then $\forall l \in \mathbb{N}$ the set $B_{\frac{1}{l}}(\xi) \cap S^{n-1}$ contains a point ξ_l from Ξ_π. But then $\xi_l \to \xi$ as $l \to \infty$, hence $\xi \in \Xi_\pi$, and $\xi \in \Xi_0 \cap \Xi_\pi$.

We shall consider two cases.

(1) There exists $\xi_0 \in S^{n-1}$ such that $(\Xi_0 \cap \Xi_\pi) \cap \xi_0^{\perp} = \varnothing$.
(2) For every $\xi \in S^{n-1}$ we have $(\Xi_0 \cap \Xi_\pi) \cap \xi^{\perp} \neq \varnothing$.

In the first case we use $S^{n-1} = \Xi_0 \cup \Xi_\pi$ to write

$$S^{n-1} = (\Xi_0 \setminus \Xi_\pi) \cup (\Xi_0 \cap \Xi_\pi) \cup (\Xi_\pi \setminus \Xi_0), \tag{8}$$

in order to conclude that

$$\xi_0^{\perp} \subset (\Xi_0 \setminus \Xi_\pi) \cup (\Xi_\pi \setminus \Xi_0). \tag{9}$$

Since

$$(\Xi_0 \setminus \Xi_\pi) \cap (\Xi_\pi \setminus \Xi_0) = \varnothing, \tag{10}$$

relation (9) yields

$$\xi_0^{\perp} \subset (\Xi_0 \setminus \Xi_\pi) \quad \text{or } \xi_0^{\perp} \subset (\Xi_\pi \setminus \Xi_0), \tag{11}$$

(we refer the reader to the end of the proof, where we show the validity of (11)). Thus, using (7) and (11) we obtain $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta) \forall \theta \in S^{n-1}$.

Consider the second case. We claim that

$$S^{n-1} = \bigcup_{\xi \in (\Xi_0 \cap \Xi_\pi)} \xi^{\perp}, \tag{12}$$

(hence, f and g are *even* and we are done). If (12) is not true, then there exists $w \in S^{n-1} \setminus \bigcup_{\xi \in (\Xi_0 \cap \Xi_\pi)} \xi^{\perp}$. But then, $w^{\perp} \cap (\Xi_0 \cap \Xi_\pi) = \varnothing$, (for, if some $\theta \in w^{\perp} \cap (\Xi_0 \cap \Xi_\pi)$ then $w \in \theta^{\perp}$ yields $w \in \bigcup_{\xi \in (\Xi_0 \cap \Xi_\pi)} \xi^{\perp}$) a contradiction.

It remains to show (11). If it is not true, then

$$\xi_0^{\perp} \cap (\Xi_0 \setminus \Xi_\pi) \neq \varnothing, \quad \text{and} \quad \xi_0^{\perp} \cap (\Xi_\pi \setminus \Xi_0) \neq \varnothing.$$

Take any $w_1 \in \xi_0^{\perp} \cap (\Xi_0 \setminus \Xi_\pi)$ and $w_2 \in \xi_0^{\perp} \cap (\Xi_\pi \setminus \Xi_0)$ and consider a big circle $E \subset \xi_0^{\perp}$ containing w_1 and w_2. Rotating if necessary we can assume that

$$E = \{w = w(t) \in S^{n-1} : \ w(t) = (\cos t, \sin t, 0, \dots, 0),\ t \in [0, 2\pi]\},$$

and

$$w_1 = (\cos t_1, \sin t_2, 0, \dots, 0), \qquad w_2 = (\cos t_2, \sin t_2, 0, \dots, 0),$$

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for some $t_1, t_2 \in [0, 2\pi]$, $t_1 < t_2$. Now put

$$t^* = \sup\{t \in [t_1, t_2) : w(t) \in \xi_0^\perp \cap (\Xi_0 \setminus \Xi_\pi)\}, \qquad w^* = w(t^*).$$

We have two possibilities,

$$\text{(a) } w^* \in \xi_0^\perp \cap (\Xi_0 \setminus \Xi_\pi), \qquad \text{(b) } w^* \in \xi_0^\perp \cap (\Xi_\pi \setminus \Xi_0).$$

If (a) is true, then $w^* \in \xi_0^\perp \cap (\Xi_\pi \setminus \Xi_0)$ since $w(t) \in \xi_0^\perp \cap (\Xi_\pi \setminus \Xi_0)$ for all $t > t^*$, and $\xi_0^\perp \cap \Xi_\pi$ is closed. But then,

$$w^* \in (\Xi_0 \setminus \Xi_\pi) \cap (\Xi_\pi \setminus \Xi_0), \tag{13}$$

which contradicts (10).

If (b) is true, then $\forall l \in \mathbb{N} \exists t_l \in [t^* - \frac{1}{l}, t^*)$ such that $w_l = w(t_l) \in \xi_0^\perp \cap (\Xi_0 \setminus \Xi_\pi)$, (otherwise $\exists l$ such that $\forall t \in [t^* - \frac{1}{l}, t^*]$ we have $w(t) \notin \xi_0^\perp \cap (\Xi_0 \setminus \Xi_\pi)$, and t^* is not a supremum). Since $w_l \to w^*$ as $l \to \infty$ and $\xi_0^\perp \cap \Xi_0$ is closed, we again have (13) which contradicts (10), and (11) is proved.

The proof of the lemma is finished. □

4. Auxiliary results used in the proof of Lemma 2

Lemma 6. *Let $n = 3$ and let f and g be two positive continuous functions on S^2. Then Σ, defined as in Lemma 2, is closed.*

Proof. We can assume that Σ is not empty. We recall that the constant is independent of $\xi \in \Sigma$, since any two great sub-spheres of S^2 intersect.

Let $(\xi_l)_{l=1}^\infty$ be a sequence of elements of non-empty Σ converging to $\xi \in S^2$, as $l \to \infty$, and let θ be any point on $\xi^\perp$. If $(\theta_l)_{l=1}^\infty$, $\theta_l \in \xi_l^\perp$, is a sequence of points converging to θ as $l \to \infty$, (the existence of such a sequence can be shown exactly as in the proof of Lemma 5), then (4) holds with θ_l instead of θ and $\xi_l^\perp$ instead of $\xi^\perp$ for all $l = 1, 2, \dots$. By continuity, $\xi \in \Sigma$. □

To formulate the next lemma we introduce some notation.

For a fixed *right-hand rule orientation* in $\mathbb{R}^3$ and a fixed direction $\xi \in S^2$ we let $\{\phi_\xi\}$ stand for the *set* of all *counter-clockwise* rotations $\phi_\xi \in SO(2)$ in $\xi^\perp$ such that

$$f(\phi_\xi(\theta)) = g(\theta) \quad \forall \theta \in \xi^\perp. \tag{14}$$

We will write $\alpha\pi \in \{\phi_\xi\}$, $\alpha \in \mathbb{R}$, meaning that the matrix of the rotation corresponding to the angle $\alpha\pi$ belongs to $\{\phi_\xi\}$. We denote

$$\mathcal{F}_\alpha := \{\xi \in S^2 : \alpha\pi \in \{\phi_\xi\}\}, \quad \alpha \in \mathbb{R}, \tag{15}$$

(in the three-dimensional case, $\mathcal{F}_0 = \Xi_0$, $\mathcal{F}_1 = \Xi_\pi$, cf. (1) and (2)).

Lemma 7. *Let $n = 3$ and let $\alpha \in \mathbb{R}$. Then, $\mathcal{F}_\alpha$ is closed.*

Proof. We can assume that $\mathcal{F}_\alpha$ is not empty.

Let $(\xi_l)_{l=1}^\infty$ be a sequence of elements of $\mathcal{F}_\alpha$ converging to $\xi \in S^2$, as $l \to \infty$, and let θ be any point on $\xi^\perp$. Consider a sequence $(\theta_l)_{l=1}^\infty$ of points $\theta_l \in \xi_l^\perp$ converging to θ as $l \to \infty$, (the existence of such a sequence can be shown exactly as in the proof of Lemma 5). By the definition of $\mathcal{F}_\alpha$, we see that

$$f(\phi_{\xi_l}(\theta_l)) = g(\theta_l) \quad \theta_l \in \xi_l^\perp, \ l \in \mathbb{N}. \tag{16}$$

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Moreover, by *Rodrigues' rotation formula*, [7], we have

$$\Phi_l(\theta_l) = \theta_l \cos(\alpha\pi) + (\xi_l \times \theta_l)\sin(\alpha\pi) + \xi_l(\xi_l \cdot \theta_l)(1 - \cos(\alpha\pi)), \tag{17}$$

where $\Phi_l = \Phi_{l,\alpha} \in SO(3)$ is a rotation around ξ_l by an angle $\alpha\pi$, and $\xi_l \times \theta_l$, $\xi_l \cdot \theta_l (=0)$, are usual vector and scalar products in $\mathbb{R}^3$. Since the restriction of Φ_l onto $\xi_l^{\perp}$ coincides with the rotation in $\xi_l^{\perp}$ by $\alpha\pi$, we see that (16) yields

$$f(\Phi_l(\theta_l)) = g(\theta_l) \quad \forall l \in \mathbb{N}. \tag{18}$$

Let $\Phi \in SO(3)$ be a rotation around ξ by an angle $\alpha\pi$. Passing to the limit in (17), and using the Rodrigues' formula again, we obtain

$$\lim_{l\to\infty} \Phi_l(\theta_l) = \Phi(\theta). \tag{19}$$

Hence, using the continuity of f and g and (19) we may pass to the limit in (18) to obtain $f(\Phi(\theta)) = g(\theta)$. Finally, due to the facts that the restriction of Φ onto $\xi^{\perp}$ coincides with the rotation by $\alpha\pi$ in $\xi^{\perp}$, and the choice of $\theta \in \xi^{\perp}$ was arbitrary, we obtain (14) with $\{\phi_\xi\} \ni \alpha\pi$. Thus, $\xi \in \mathcal{F}_\alpha$, and the result follows. □

The following lemma is a well-known consequence of the Baire category theorem, we include it here for the convenience of the reader.

We recall that a set A is called *nowhere dense* in a topological space Y, if the closure of A has an empty interior, [11, p. 42]. The Baire category theorem claims that *no complete metric space can be written as a countable union of nowhere dense sets*; see [11, p. 43].

Lemma 8. *Let $\overline{\mathcal{B}_\beta}(\theta) \subset S^2$ be the spherical geodesic closed ball centered at $\theta \in S^2$ of radius $\beta\pi$, $\beta > 0$, and let $\overline{\mathcal{B}_\beta}(\theta) = \bigcup_{k=1}^{\infty} F_k$, where all F_k are closed. Then, there exists $k_o \in \mathbb{N}$ such that* $\operatorname{int}(F_{k_o}) \neq \varnothing$.

Proof. It is enough to observe that $\overline{\mathcal{B}_\beta}(\theta)$ is a complete metric space, since it is a compact subset of the complete metric space S^2 with the usual metric of S^2. Since F_k are all closed, the result follows from the Baire category theorem. □

The next result is a consequence of the properties of the *Funk transform*, [6, Chapter III, Section 1],

$$Rf(\xi) = \int_{\xi^{\perp}} f(\theta)d\theta, \quad \xi \in S^{n-1}.$$

Here $d\theta$ is the Lebesgue measure on $\xi^{\perp}$.

Lemma 9. *Let $n = 3$ and let f and g be as in Theorem 1. Then,*

$$f_e(\theta) = g_e(\theta) \quad \forall \theta \in S^2. \tag{20}$$

Proof. Let $\xi \in S^2$, and let ϕ_ξ be the corresponding rotation in $\xi^{\perp}$. By the rotation invariance of the Lebesgue measure on $\xi^{\perp}$, we have

$$\int_{\xi^{\perp}} f(\phi_\xi(\theta))d\theta = \int_{\xi^{\perp}} f(\theta)d\theta.$$

Hence,

$$Rf(\xi) = Rg(\xi), \quad \forall \xi \in S^2,$$

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and we obtain (20), (to see the validity of the last statement, apply Theorem C.2.4 from [2, p. 430] to $f_e - g_e$). $\square$

To formulate the last auxiliary statement we introduce some more notation.

Let $\alpha \in (0,1)$ and let $\mathbf{S}_1, \mathbf{S}_2$ be any two spherical circles in the standard metric of S^2, both of radius $\alpha\pi$. The union $\mathfrak{l} \cup \mathfrak{m}$ of two *open* arcs $\mathfrak{l} \subset \mathbf{S}_1$ and $\mathfrak{m} \subset \mathbf{S}_2$ will be called a *spherical X-figure* if the angle between arcs is in $(0, \frac{\pi}{4})$, the length of the arcs is less than $\alpha\pi$, and the arcs intersect at their centers only, $\mathfrak{l} \cap \mathfrak{m} = \{w\}$. The point w will be called the *center* of the X-figure. The ends of the arcs of the X-figure will be called the *vertices* of X.

Let f be a function on S^2, and let w be a center of a spherical X-figure. If for every $u \in X$ we have $f(u) = f(w)$, we will write $\exists X_{f(w)} \subset S^2$.

Lemma 10. *Let $n = 3$ and let f and g be two continuous functions satisfying the conditions of Theorem 1. Assume also that $\mathcal{F}_\alpha$, defined by (15) for some $\alpha \in (0,1)$, satisfies $\operatorname{int}(\mathcal{F}_\alpha) \neq \varnothing$. If $\xi \in \operatorname{int}(F_\alpha)$, then*

$$\forall w \in \xi^\perp \quad \exists\, X_{f_e(w)} \subset S^2, \tag{21}$$

and one of the arcs of $X_{f_e(w)}$ is orthogonal to $\xi^\perp$. Moreover,

$$\forall w, \theta \in \xi^\perp \quad \exists X_{f_e(w)},\ X_{f_e(\theta)} \in S^2 :\ \Theta(X_{f_e(w)}) = X_{f_e(\theta)}, \tag{22}$$

where $\Theta \in SO(3)$ is such that $\Theta(\xi) = \xi$ and $\Theta(w) = \theta$.

The proof of this lemma is technical and we give it in the Appendix.

5. Proof of Lemma 2

The proof will be split into two lemmata proved below. We start with a few comments.

By Lemma 1 we can assume $\Sigma \neq \varnothing$. If $F := S^2 \setminus (\Xi_0 \cup \Xi_\pi \cup \Sigma) \neq \varnothing$, we consider two cases:

$$(1)\ \exists \xi \in F : \{\phi_\xi\} \ni \alpha\pi, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}; \tag{23}$$

$$(2)\ F = \bigcup_{r \in \mathbb{Q}} F_r, \quad F_r := F \cap \mathcal{F}_r, \tag{24}$$

where $\mathcal{F}_r$ is defined as in (15) with r instead of α, and $\mathbb{Q}$ is the set of rational numbers r such that $r = \frac{p}{q}$ for co-prime integers p and $q \neq 0$.

To prove Lemma 2 it is enough to show that both (23) and (24) are not possible.

Lemma 11. *Let $n = 3$, and let f, g, Σ, F be as above. Then (23) is not possible.*

Proof. Let ϕ_α stand for the rotation in $\xi^\perp$ through the angle $\alpha\pi$. Relations (20) and (14) give

$$f_e(\phi_\alpha(\theta)) = g_e(\theta) = f_e(\theta) \quad \forall \theta \in \xi^\perp.$$

Hence, we have

$$f_e(\phi_{2\alpha}(\theta)) = g_e(\phi_\alpha(\theta)) = g_e(\theta) = f_e(\theta) \quad \forall \theta \in \xi^\perp.$$

After iteration we obtain

$$f_e(\phi_{m\alpha}(\theta)) = f_e(\theta) \quad \forall m \in \mathbb{N}\ \forall \theta \in \xi^\perp.$$

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If for some $\xi \in F$ the set $\{\phi_\xi\}$ contains the angle $\alpha\pi$ for an irrational α, then the points $\phi_{m\alpha}(\theta), m \in \mathbb{N}$, form a dense set on $\xi^\perp$, [8]. By continuity, $f_e(\theta) = \text{const}$ for all $\theta \in \xi^\perp$, which contradicts the fact that $\xi \notin \Sigma$. □

Lemma 12. *Let $n = 3$, and let f, g, Σ, F be as above. Then (24) is not possible.*

Proof. We claim at first that

$$\exists r_o \in \mathbb{Q}: \quad \operatorname{int}(F_{r_o}) \neq \varnothing, \tag{25}$$

where $\mathbb{Q}$ is as in (24).

Indeed, assume that for all $r \in \mathbb{Q}$ we have $\operatorname{int}(F_r) = \varnothing$. By Lemmata 5 and 6, the set F is *open* as a compliment of the union of three closed sets. Hence, there exist $\beta > 0$ sufficiently small and $\theta \in F$ such that the spherical geodesic closed ball $\overline{\mathcal{B}_\beta}(\theta)$ (centered at θ of radius $\beta\pi$) is contained in F. Moreover, using (24) we can write

$$\overline{\mathcal{B}_\beta}(\theta) = \bigcup_{r \in \mathbb{Q}} (\overline{\mathcal{B}_\beta}(\theta) \cap \mathcal{F}_r), \quad \forall r \in \mathbb{Q}, \tag{26}$$

where $\overline{\mathcal{B}_\beta}(\theta) \cap \mathcal{F}_r$ are all closed (apply Lemma 7 with $\alpha = r$). Now our assumption together with $(\overline{\mathcal{B}_\beta}(\theta) \cap \mathcal{F}_r) \subset F_r$ yields

$$\operatorname{int}(\overline{\mathcal{B}_\beta}(\theta) \cap \mathcal{F}_r) \subset \operatorname{int}(F_r) = \varnothing \quad \forall r \in \mathbb{Q}. \tag{27}$$

We see that (26) and (27) contradict the Baire category theorem, since $\mathbb{Q}$ is countable, apply Lemma 8 with F_k as suitably enumerated $\overline{\mathcal{B}_\beta}(\theta) \cap \mathcal{F}_r$. Thus, (25) holds.

Changing the direction of rotation if necessary, and using the fact that our rotations are 2π-periodic, we can assume that $0 < r_o < 1$.

Let $\xi \in \operatorname{int}(F_{r_o})$. Then, since $\xi \notin \Sigma$, and $f_e = g_e$ on S^2, f_e is not a constant function of $\xi^\perp$. We will show that the last statement is impossible, thus getting a contradiction. The idea is to use Lemma 10 (since $F_{r_o} \subseteq \mathcal{F}_{r_o}$ we can apply it with $\alpha = r_o$) to show the existence of an uncountable family of *disjoint* spherical $X_{f_e(w)}$-figures, $w \in \xi^\perp$, and then to use the fact that such family does not exist.

Define

$$\mathcal{M} = \min_{w \in \xi^\perp} f_e(w), \qquad \mathfrak{M} = \max_{w \in \xi^\perp} f_e(w).$$

Since, by assumption, f_e is not constant on $\xi^\perp$, the interval $[\mathcal{M}, \mathfrak{M}]$ is of capacity continuum. By the intermediate value theorem, for every $y \in [\mathcal{M}, \mathfrak{M}]$ there exists $w \in \xi^\perp$ such that $f_e(w) = y$. Denote $B_y = \{w \in \xi^\perp : f_e(w) = y\}$ and let w_y be any fixed element of B_y. Define

$$\mathfrak{A} = \{w_y\}_{y \in [\mathcal{M}, \mathfrak{M}]} \subset \xi^\perp.$$

Since between elements of the sets $\mathfrak{A}$ and $[\mathcal{M}, \mathfrak{M}]$ there is a one-to-one correspondence, we see that the capacity of $\mathfrak{A}$ is continuum as well.

By (21) of Lemma 10 (with $\alpha = r_o$) $\forall w \in \xi^\perp\, \exists X_{f_e(w)}$, and we define the set

$$\mathfrak{B} = \{X_{f_e(w_y)}\}_{y \in [\mathcal{M}, \mathfrak{M}]}$$

of all spherical X-figures with centers at $w_y \in \mathfrak{A}$. This set is again of capacity continuum since there is a one-to-one correspondence between the X-figures and their centers. Moreover, due to the fact that on each $X_{f_e(w_y)}$-figure the function f_e takes a constant value $f_e(w_y)$, and

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$f_e(w_{y_1}) \neq f_e(w_{y_2})$, provided $y_1 \neq y_2$, (we chose the unique w_y from each B_y), we see that all X-figures from $\mathfrak{B}$ are pairwise disjoint.

Thus, if $f_e \neq$ const on $\xi^\perp$, $\exists\mathfrak{B}$, which is an *uncountable* family of *disjoint* X-figures. It remains to show that

$$\mathfrak{B} \quad \text{does not exist.} \tag{28}$$

To show (28), we will use Lemma 10 and some elementary geometry.

We observe at first, that we can assume that all vertices of all $X \in \mathfrak{B}$ are located on the parallels $P_{\pm,\delta}(\xi) := \{\theta \in S^2 : \theta \cdot \xi = \pm\delta\}$ for some small $\delta > 0$. Indeed, by Lemma 10 we know that one of the arcs of each of the figures is orthogonal to $\xi^\perp$ and by the definition of the X-figure we know that the angle between the arcs is in $(0, \frac{\pi}{4})$. Therefore, using the fact that (by (22) of Lemma 10) all figures are rotations of each other around ξ, and considering the family

$$\mathfrak{B}^* := \{X_{f_e(w_y)} \cap E_\delta(\xi)\}_{y\in[\mathcal{M},\mathfrak{M}]}$$

instead of $\mathfrak{B}$, $E_\delta(\xi) := \{\theta \in S^2 : |\theta \cdot \xi| \leq \delta\}$, it is enough, by taking δ small enough, to show that $\mathfrak{B}^*$ does not exist; the observation follows.

Second, we "separate" points from $\mathfrak{A}$. To do this we let w_{y_1} be any element of $\mathfrak{A}$. We claim that

$$d := \mathrm{dist}_{S^2}(w_{y_1}, \mathfrak{A} \setminus \{w_{y_1}\}) = \inf_{y\in([\mathcal{M},\mathfrak{M}]\setminus\{y_1\})} \mathrm{dist}_{S^2}(w_{y_1}, w_y) > 0. \tag{29}$$

Assume that $d = 0$. By the definition of the infimum, for arbitrarily small $\epsilon > 0$, there exist $w_{y_2} \in (\mathfrak{A} \setminus \{w_{y_1}\})$: $\mathrm{dist}_{S^2}(w_{y_1}, w_{y_2}) < \epsilon$, and by Lemma 10 we know that $X_{f_e(w_{y_1})}, X_{f_e(w_{y_2})}$ are rotations of each other. We can also assume, that the figure $X_{f_e(w_{y_2})}$ is obtained by a counterclockwise rotation of $X_{f_e(w_{y_1})}$, (the case of the clockwise rotation is similar). Now, consider an auxiliary spherical X-figure, X_a, centered at $w \in \xi^\perp$, obtained by a counterclockwise rotation of $X_{f_e(w_{y_1})}$ around ξ such that the upper left vertex of X_a is the upper right vertex of $X_{f_e(w_{y_1})}$, (the parallels $P_{\pm,\delta}(\xi)$ are invariant under the rotation, so the vertices of X_a are on $P_{\pm,\delta}(\xi)$). Since the angle between arcs in our X-figures is positive, we have $\mathrm{dist}_{S^2}(w_{y_1}, w) > 0$, and w_{y_1}, w_{y_2} must satisfy

$$\epsilon > \mathrm{dist}_{S^2}(w_{y_1}, w_{y_2}) \geq \mathrm{dist}_{S^2}(w_{y_1}, w), \tag{30}$$

(otherwise, by the choice of X_a, and the fact that the vertices of the figures $X_{f_e(w_{y_1})}, X_{f_e(w_{y_2})}$ are on $P_{\pm,\delta}(\xi)$, we have $X_{f_e(w_{y_1})} \cap X_{f_e(w_{y_2})} \neq \emptyset$). But we could pick ϵ such that $\epsilon < \mathrm{dist}_{S^2}(w_{y_1}, w)$, which contradicts (30).

Thus, $d > 0$, and we have a disjoint uncountable family of sub-arcs of $\xi^\perp$ centered at w_y, each of length $\frac{d}{5}$. This is impossible, since $\xi^\perp$ is of finite length, and (28) follows.

We see that our assumption that f_e is not constant on $\xi^\perp$ (which was a consequence of the assumption $\mathrm{int}(F_{r_o}) \neq \emptyset$) is wrong, and $\mathrm{int}(F_{r_o}) = \emptyset$. Hence, (24) is impossible. This finishes the proof of Lemma 12. □

Finally, by Lemmata 11 and 12 we have $F = \emptyset$, and the proof of Lemma 2 is finished.

6. Concluding remarks

We start with some remarks about analogues of Theorems 1–3 in the two-dimensional case. In this case $\xi^\perp$ consists of a pair of antipodal points on S^1, and the action of the group of rotations $SO(2)$ on $\xi^\perp$ is reduced to a reflection.

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An analogue of Theorem 1 (n = 2). *Let f and g be two continuous functions on S^1 such that for every $\xi \in S^1$ we have $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for every $\theta \in \xi^\perp$. Then it is not true that $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^1$.*

To show this, we divide S^1 into four open arcs A_j of equal length such that $A_1 = -A_3$, $A_2 = -A_4$, and $\sum_{j=1}^4 |A_j| = 2\pi$. Then, we define

$$f = f_1\chi_{A_1} - f_2\chi_{A_2} - f_3\chi_{A_3} + f_4\chi_{A_4},$$
$$g = f_1\chi_{A_1} + f_2\chi_{A_2} - f_3\chi_{A_3} - f_4\chi_{A_4}.$$

Here χ_{A_j} are the characteristic functions of the corresponding arcs, and f_j are continuous functions on S^1 that are positive inside A_j, vanishing at their ends, and $f_2, f_4 : f_2(\theta)\chi_{A_2}(\theta) = f_4(-\theta)\chi_{A_4}(-\theta) \forall\theta \in A_2$.

By definition, we have $f = g$ on $A_1 \cup A_3$ and $f(\theta) = g(-\theta) \forall\theta \in A_2 \cup A_4$, since $A_4 = -A_2$. On the other hand, we see that $f = g$ does not hold, since $f(\theta) = -g(\theta) \forall\theta \in A_2$. Moreover, since $A_3 = -A_1$, $A_4 = -A_2$, we have

$$\begin{aligned} g(-\theta) &= f_1(-\theta)\chi_{A_1}(-\theta) + f_2(-\theta)\chi_{A_2}(-\theta) - f_3(-\theta)\chi_{A_3}(-\theta) - f_4(-\theta)\chi_{A_4}(-\theta) \\ &= -f_3(-\theta)\chi_{A_1}(\theta) - f_4(-\theta)\chi_{A_2}(\theta) + f_1(-\theta)\chi_{A_3}(\theta) + f_2(-\theta)\chi_{A_4}(\theta). \end{aligned}$$

Since f_1, f_3 are positive on A_1, A_3, $-f_3(-\theta)\chi_{A_1}(\theta) \neq f_1(\theta)\chi_{A_1}(\theta)$, and $f(\theta) = g(-\theta)$ for all $\theta \in S^1$ does not hold either.

Analogues of Problems 1 and 2 in the case $n = 2$ are known to have a negative answer as well.

An analogue of Problem 1 (n = 2). *Suppose that K and L are convex bodies in $\mathbb{R}^2$ and let $H = H(\xi)$ be a one-dimensional subspace of $\mathbb{R}^2$ containing $\xi^\perp$. If $K|H(\xi)$ is congruent to $L|H(\xi)$ for all $\xi \in S^1$, does it follow that K is a translate of $\pm L$?*

Since the projections are segments, the congruence of the projections is reduced to a translation. Moreover, due to the fact that for every $\theta \in \xi^\perp$ we have

$$\text{length}(K|H) = h_K(\theta) + h_K(-\theta) = h_L(\theta) + h_L(-\theta) = \text{length}(L|H),$$

to construct a counterexample it is enough to consider a body K of *constant width* that is not a disc, i.e., K such that $h_K(\theta) + h_K(-\theta) = 2w \forall\theta \in S^1$, but $h_K \not\equiv w$, and the disc L of radius w; see [1, Section 15, p. 135], and [2, p. 109].

An analogue of Problem 2 (n = 2). *Suppose that K and L are convex bodies in $\mathbb{R}^2$ and let $H = H(\xi)$ be a one-dimensional subspace of $\mathbb{R}^2$ containing $\xi^\perp$. If $K \cap H(\xi)$ is congruent to $L \cap H(\xi)$ for all $\xi \in S^1$, does it follow that K is a translate of $\pm L$?*

We observe that for every $\theta \in \xi^\perp$,

$$\text{length}(K \cap H) = \rho_K(\theta) + \rho_K(-\theta) = \rho_L(\theta) + \rho_L(-\theta) = \text{length}(L \cap H),$$

and for a counterexample one can take a plane *equichordal* body K that is not a disc, i.e., K such that $\rho_K(\theta) + \rho_K(-\theta) = 2w \forall\theta \in S^1$, but $\rho_K \not\equiv w$, and the disc L of radius w; see [2, p. 255, Theorem 6.3.2, and p. 276].

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Finally, we would like to mention some open questions.

1. *Let f and g be two continuous functions on S^3, and let their restrictions to any two-dimensional great sub-sphere E of S^3 coincide after some rotation $\phi_E \in SO(3)$ of this sphere, $f(\phi_E(\theta)) = g(\theta) \forall \theta \in E$. Is it true that $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^3$?*

   Some results in this direction are implicitly contained in [5, Chapter 3].
2. *Is it possible to relax the continuity assumption in* Theorem 1, *say, to the class of bounded measurable functions on the unit sphere?*

   This seems to be possible. If not, it would be interesting to find a counterexample to the analogue of Lemma 1.
3. *Let $n \geq 3$ and let K and L be two convex bodies in $\mathbb{R}^n$ containing the origin in their interior. Assume also that for every two-dimensional subspace H there exists $\psi = \psi_H \in SO(2)$ such that the projections of the bodies K and L onto H satisfy $\psi(K|H) \subseteq L|H$. Is it true that $K \subseteq L$ or $K \subseteq -L$?*

In this connection, see the results of Daniel Klain, [9], who gave a negative answer to the following question.

Consider two compact convex subsets K and L of $\mathbb{R}^n$. Suppose that, for a given dimension $1 \leq d < n$, every d-dimensional orthogonal projection (shadow) of L contains a translate of the corresponding projection of K. Does it follow that the original set L contains a translate of K?

A question, similar to 3, can be asked about sections of star bodies.

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Appendix. Proof of Lemma 10

The result is a consequence of three propositions. To formulate the first one we will introduce some notation.

Take any $\xi \in \operatorname{int}(\mathcal{F}_\alpha)$ and any $w_1 \in \xi^\perp$. Rotating if necessary, we can assume that $\xi = (0, 0, 1)$. Let w_2 be a unit vector in $\xi^\perp$ obtained by the rotation of w_1 through the angle $\alpha\pi$, $\alpha \in (0, 1)$, where the direction of the rotation is determined by ξ. We shall assume that w_2 is obtained from w_1 by a counterclockwise rotation, (the case of the clockwise rotation is similar). We denote by $\mathbf{S}(w_1, \alpha) \subset S^2$ the *spherical circle* with center w_1 and radius $\alpha\pi$ in the standard metric of the unit sphere.

Proposition 1. *Let $\mathcal{F}_\alpha$ be as in Lemma 10, and let ξ, w_1 and w_2 be as above. Then, there is an arc $\mathfrak{l}_1 \subset \mathbf{S}(w_1, \alpha)$, $\mathfrak{l}_1 \ni w_2$, (see Fig. 1), such that $\forall u \in \mathfrak{l}_1$,*

$$f_e(u) = f_e(w_1). \tag{31}$$

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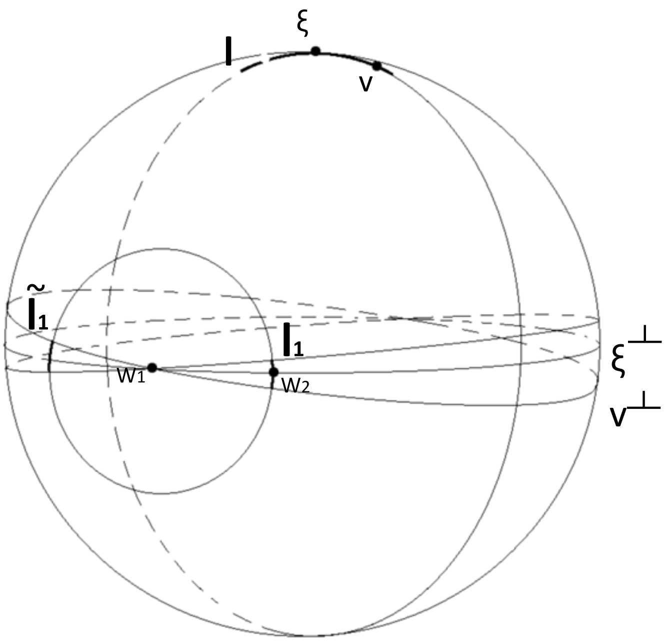

Fig. 1. Arc l_1 containing $w_2 \in \xi^{\perp}$.

Proof. Since $\xi \in \operatorname{int}(\mathcal{F}_\alpha)$, there exist $m_o \in \mathbb{N}$ and a Euclidean ball $B_{\frac{1}{m_o}}(\xi)$ such that $B_{\frac{1}{m_o}}(\xi) \cap S^2 \subset \operatorname{int}(\mathcal{F}_\alpha)$. Taking $m > 2\max(\frac{1}{\alpha\pi}, \frac{1}{(1-\alpha)\pi}, m_o)$ we can assume that $B_{\frac{1}{m}}(\xi) \cap S^2 \subset \operatorname{int}(\mathcal{F}_\alpha)$.

Let $\mathfrak{l} = \mathfrak{l}_\xi = B_{\frac{1}{4m}}(\xi) \cap w_1^{\perp}$ be a small arc centered at ξ. For every point $v \in \mathfrak{l}_\xi$ we consider $v^{\perp}$, and observe that w_1 belongs to $v^{\perp}$ for all $v \in \mathfrak{l}_\xi$. Next, we define the set

$$A_\xi := \mathbf{S}(w_1, \alpha) \cap \bigcup_{v \in \mathfrak{l}_\xi} v^{\perp} \tag{32}$$

consisting of two arcs of $\mathbf{S}(w_1, \alpha)$, the left one, $\widetilde{\mathfrak{l}}_1$, and the right one, $\mathfrak{l}_1$, (see Fig. 1). Since $w_2 \in \xi^{\perp}$, $\operatorname{dist}_{S^2}(w_1, w_2) = \alpha\pi$, and $\xi \in \mathfrak{l}_\xi$ we see that $w_2 \in \mathfrak{l}_1$.

It remains to check (31). To this end, we take any $u \in \mathfrak{l}_1$. By the definition of the set A_ξ there exists $v \in \mathfrak{l}_\xi$ such that $u \in v^{\perp}$. Since $v \in \mathcal{F}_\alpha$ and $\operatorname{dist}_{S^2}(u, w_1) = \alpha\pi$, there exists a rotation $\phi_v \in SO(2)$ such that

$$u = \phi_v(w_1). \tag{33}$$

We remind that by (20) of Lemma 9 we have $g_e(\theta) = f_e(\theta) \forall \theta \in S^2$. Hence, (33) and $f_e(\phi_v(w_1)) = g_e(w_1)$ yield

$$f_e(u) = f_e(\phi_v(w_1)) = g_e(w_1) = f_e(w_1).$$

This gives (31) and the proposition follows. $\square$

Proposition 2. *Let $\mathcal{F}_\alpha$ be as in Lemma 10 and let $\xi, w_1, w_2, \mathfrak{l}_1$ be as in Proposition 1. Then (21) holds.*

Proof. We will show that for every $w_1 \in \xi^{\perp}$ there exists a spherical X-figure, centered at w_1, that is a union of two arcs $\mathfrak{l}_2 \cup \mathfrak{l}_3$ such that (31) holds for all $u \in \mathfrak{l}_2 \cup \mathfrak{l}_3$. Moreover, we will prove that $\mathfrak{l}_2$ is orthogonal to $\xi^{\perp}$, (see Fig. 2).

The proof is, essentially, the double repetition of the argument from the proof of Proposition 1. We start with the construction of $\mathfrak{l}_3$, (see Fig. 2).

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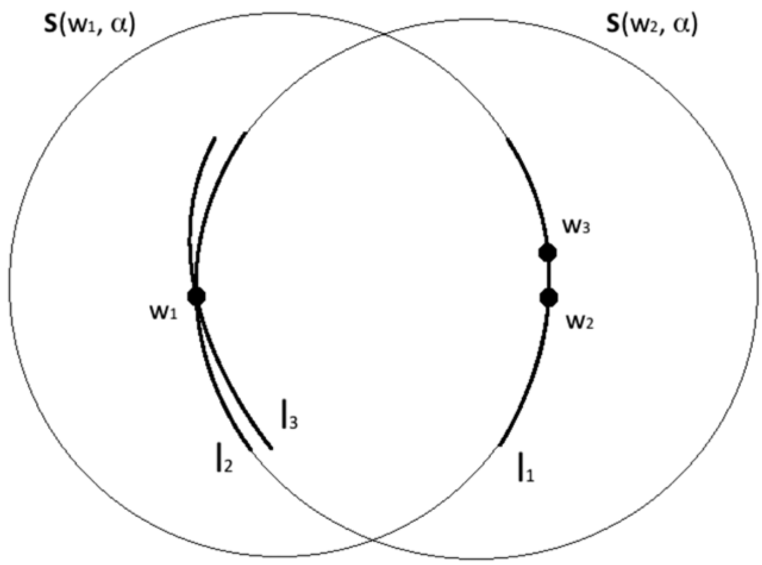

Fig. 2. Arcs l_1, l_2 and l_3.

We take any point $w_3 \in \mathfrak{l}_1$, $w_3 \neq w_2$, and denote by $\eta^{\perp}$, $\eta \in S^2$, a big circle containing w_1 and w_3. Since $\eta^{\perp} \cap \xi^{\perp} \ni w_1$, and $\operatorname{dist}_{S^2}(u, w_1) = \alpha\pi$ for all $u \in \mathfrak{l}_1$, it is readily seen that $\eta^{\perp} \to \xi^{\perp}$ (and $\eta \to \xi$) as $w_3 \to w_2$ along $\mathfrak{l}_1$. Hence, we can take w_3 so close to w_2 that $\eta \in B_{\frac{1}{4m}}(\xi) \cap S^2$, where m is chosen as in the previous proposition.

Now we repeat the part of the proof of the previous proposition with η instead of ξ and w_3 instead of w_1.

Let $\mathfrak{l}_\eta = B_{\frac{1}{4m}}(\eta) \cap w_3^{\perp}$ be a small arc centered at η. Since $\eta \in B_{\frac{1}{4m}}(\xi) \cap S^2$, we have $\mathfrak{l}_\eta \subset B_{\frac{1}{m}}(\xi) \cap S^2 \subset \operatorname{int}(\mathcal{F}_\alpha)$.

For every point $v \in \mathfrak{l}_\eta$ we consider $v^{\perp}$, and observe that w_3 belongs to $v^{\perp}$ for all $v \in \mathfrak{l}_\eta$. Next, similar to (32), we define the set

$$A_\eta := \mathbf{S}(w_3, \alpha) \cap \bigcup_{v \in \mathfrak{l}_\eta} v^{\perp}, \tag{34}$$

consisting of two arcs of $\mathbf{S}(w_3, \alpha)$, the left and the right ones. This time we choose the left one, this is our $\mathfrak{l}_3$. Observe that since $\operatorname{dist}_{S^2}(w, w_3) = \alpha\pi$ for all $w \in \mathfrak{l}_3$, $\operatorname{dist}_{S^2}(w_1, u) = \alpha\pi$ for all $u \in \mathfrak{l}_1$, and $\operatorname{dist}_{S^2}(w_1, w_2) = \operatorname{dist}_{S^2}(w_1, w_3) = \alpha\pi$, $w_3 \in \mathfrak{l}_1$, we have $w_1 \in \mathfrak{l}_3$.

We claim that (31) holds for all $u \in \mathfrak{l}_3$.

Take any $u \in \mathfrak{l}_3$. Since $\mathfrak{l}_3 \subset A_\eta$, (34) yields the existence of $v \in \mathfrak{l}_\eta$ such that $u = \mathfrak{l}_3 \cap v^{\perp}$. Moreover, since $v \in \mathfrak{l}_\eta \subset \mathcal{F}_\alpha$ and $\operatorname{dist}_{S^2}(u, w_3) = \alpha\pi$, we have $\phi_v(u) = w_3$. Applying Lemma 9 we have $g_e(\theta) = f_e(\theta) \forall \theta \in S^2$, and $f_e(\phi_v(u)) = g_e(u)$ together with $w_3 \in \mathfrak{l}_1$ yield

$$f_e(u) = g_e(u) = f_e(\phi_v(u)) = f_e(w_3). \tag{35}$$

Since $u \in \mathfrak{l}_3$ was arbitrary, we see that f_e takes the constant value $f_e(w_3)$ on $\mathfrak{l}_3$, and since $w_3 \in \mathfrak{l}_1$, by Proposition 1, we have (31) for all $u \in \mathfrak{l}_3$.

Now we construct $\mathfrak{l}_2$. We argue exactly as in the proof of Proposition 1 with w_2 instead of w_1 and with the choice of the left arc instead of the right one, (see Fig. 2).

Let $\mathfrak{l} = \mathfrak{l}_\xi = B_{\frac{1}{4m}}(\xi) \cap w_2^{\perp}$ be a small arc centered at ξ. For every point $v \in \mathfrak{l}_\xi$ we consider $v^{\perp}$, and observe that w_2 belongs to $v^{\perp}$ for all $v \in \mathfrak{l}_\xi$. Next, we define the set analogous to A_ξ from (32) with w_2 instead of w_1, consisting of two arcs of $\mathbf{S}(w_2, \alpha)$. This time we choose the

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left one. This is our $\mathfrak{l}_2$, (on Fig. 1 imagine w_2 instead of w_1, and $\widetilde{\mathfrak{l}_1} \ni w_1$). By construction, $\mathfrak{l}_2$ is orthogonal to $\xi^\perp$.

Since $w_1, w_2 \in \xi^\perp$, $\mathrm{dist}_{S^2}(w_1, w_2) = \alpha\pi$, we see that $w_1 \in \mathfrak{l}_2$. We claim that (31) holds for all $u \in \mathfrak{l}_2$.

Take any $u \in \mathfrak{l}_2$. It is readily seen that $\exists v \in \mathfrak{l}_\xi$ such that $u = \mathfrak{l}_2 \cap v^\perp$. Hence, $g_e(\theta) = f_e(\theta) \forall \theta \in S^2$ and $\phi_v(u) = w_2$ yield (35) with w_2 instead of w_3. Since $u \in \mathfrak{l}_2$ was arbitrary, we see that f_e takes the constant value $f_e(w_2)$ on $\mathfrak{l}_2$, and since $w_1 \in \mathfrak{l}_2$, we have (31) for all $u \in \mathfrak{l}_2$.

Thus, we have constructed the X-figure, which is the union of two arcs $\mathfrak{l}_2 \cup \mathfrak{l}_3$ such that $\forall u \in \mathfrak{l}_2 \cup \mathfrak{l}_3$ (31) holds. This is our $X_{f_e(w_1)}$. $\square$

Proposition 3. *Let $\mathcal{F}_\alpha$ be as in Lemma 10 and let $\xi, w_1, \mathfrak{l}_2$ and $\mathfrak{l}_3$ be as in Proposition 2. Then we have (22).*

Proof. Let $\theta_1 \in \xi^\perp, \theta_1 \neq w_1$, and let $\Theta \in SO(3)$ be the rotation leaving ξ fixed such that $\Theta(w_1) = \theta_1$. Since $\Theta(B_{\frac{1}{m}}(\xi) \cap S^2) = B_{\frac{1}{m}}(\xi) \cap S^2$, (where m is as in the proofs of Propositions 1 and 2), we have $\Theta(B_{\frac{1}{m}}(\xi)) \cap S^2 \subset \mathrm{int}(\mathcal{F}_\alpha)$. Hence, we can repeat the proofs of Propositions 1 and 2 with $f_e \circ \Theta$, $g_e \circ \Theta$ instead of f_e, g_e. Since Θ is an isometry on S^2 and $f_e \circ \Theta(w_1) = \theta_1$, we obtain $X_{f_e(\theta_1)}$, satisfying (22), $X_{f_e(\theta_1)} = \Theta(\mathfrak{l}_2) \cup \Theta(\mathfrak{l}_3)$. $\square$

Finally, Lemma 10 follows from Propositions 2 and 3.

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