A Lemma of Nakajima and Süss on Convex Bodies

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Abstract. Let K and L be two convex bodies in $\mathbb{R}^n$ such that their projections onto every $(n-1)$-dimensional subspace are translates of each other. Then K is a translate of L. We give a very simple analytic proof of this fact.

1. INTRODUCTION. In 1932, S. Nakajima [**5**] and W. Süss [**8**] independently proved (for $n = 3$) the following result.

Theorem 1. *Let K and L be two convex bodies (compact convex sets with a nonempty interior) in $\mathbb{R}^n$, for $n \geq 3$. Then K is a translate of L, provided the orthogonal projections of the bodies onto every $(n-1)$-dimensional subspace are translates of each other.*

Presently, several nontrivial *geometric* proofs of this result are known, (see, for example, the book of R. Gardner, [**2**], page 101 and notes on pages 126–127); in particular, we would like to mention a very elegant "high-school" proof, obtained by I. Lieberman, ([**1**], Lemma I, 5).

In this note, we suggest both a very simple and short *analytic* proof of Theorem 1. We use only the definition of the *support function* h_M of a compact convex set $M \subset \mathbb{R}^n$,

$$h_M(x) := \sup_{y \in M} x \cdot y, \quad \text{for} \quad x \in \mathbb{R}^n, \quad \text{with} \quad x \cdot y = x_1 y_1 + \cdots + x_n y_n. \tag{1}$$

We also use the following well-known fact from linear algebra. If f is a *linear function* in $\mathbb{R}^n$, i.e.,

$$\forall x, y \in \mathbb{R}^n, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \text{with} \quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \tag{2}$$

then there exists $a \in \mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n$, $f(x) = a \cdot x$.

2. PROOF. To begin the proof, we introduce some notation.

Let ξ be a unit vector in $\mathbb{R}^n$. We denote by $\xi^\perp$ the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to ξ,

$$\xi^\perp = \{y \in \mathbb{R}^n : y \cdot \xi = 0\}, \qquad \text{for} \quad |\xi| = 1,$$

and by $K|\xi^\perp$ the *orthogonal projection* of a convex body K onto $\xi^\perp$,

$$K|\xi^\perp := \{x \in \xi^\perp : x + \lambda\xi \in K \quad \text{for some} \quad \lambda \in \mathbb{R}\}.$$

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Using (1), it is not hard to see that, given a compact convex *set* $M \subset \mathbb{R}^n$ and an arbitrary vector $b \in \mathbb{R}^n$, we have

$$h_{M+b}(x) = h_M(x) + b \cdot x \qquad \forall x \in \mathbb{R}^n, \tag{3}$$

where $M + b = \{x \in \mathbb{R}^n : x = y + b, \quad y \in M\}$. Moreover, given a subspace $\xi^\perp$ of $\mathbb{R}^n$ and a convex body K, we have

$$h_{K|\xi^\perp}(x) = h_K(x) \qquad \forall x \in \xi^\perp. \tag{4}$$

Now we prove Theorem 1.

Proof. By condition of the theorem, for every unit vector ξ, there exists a vector $a_\xi \in \xi^\perp$ such that

$$K|\xi^\perp = L|\xi^\perp + a_\xi. \tag{5}$$

Using (3) and (5), we see that for every unit vector ξ there exists a vector $a_\xi \in \xi^\perp$ such that

$$h_{K|\xi^\perp}(x) = h_{L|\xi^\perp}(x) + a_\xi \cdot x \qquad \forall x \in \xi^\perp. \tag{6}$$

Moreover, by (4) and (6),

$$h_K(x) = h_L(x) + a_\xi \cdot x \qquad \forall x \in \xi^\perp. \tag{7}$$

Our goal is to show that

$$\exists a \in \mathbb{R}^n \quad \text{such that} \quad h_K(x) = h_L(x) + a \cdot x \qquad \forall x \in \mathbb{R}^n. \tag{8}$$

In other words, we need to show that $f := h_K - h_L$ is a linear function, i.e., f satisfies (2).

Since the linearity is a *two-dimensional* property, (8) follows immediately from (7). Indeed, let $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ be given. Consider a two-dimensional subspace $L_{x,y}$ containing x and y. It is clear that $L_{x,y} \subset \xi^\perp$ for some unit $\xi \in \mathbb{R}^n$ and (7) yields

$$f(\alpha x + \beta y) = a_\xi \cdot (\alpha x + \beta y) = \alpha a_\xi \cdot x + \beta a_\xi \cdot y = \alpha f(x) + \beta f(y). \qquad \blacksquare$$

This result does not hold in the two-dimensional case (consider a disc and a Reuleaux triangle of the same width, [**2**], page 109).

3. AN OPEN PROBLEM ABOUT CONGRUENT PROJECTIONS. It is very natural to ask what happens if the group of translations is replaced by some larger group, say, by the group of rigid motions. The following problem is still open, even in the three-dimensional case. Let K and L be two convex bodies such that their projections onto every subspace are congruent. Does it follow that there exists $a \in \mathbb{R}^3$ such that $K = \pm L + a$ (i.e., the bodies coincide up to translation and reflection)?

We refer the reader to the books of V. P. Golubyatnikov ([**3**], Chapters 1-3) and of R. J. Gardner, ([**2**], Chapters 3 and 7) for many other related problems and results; see also [**4**] and [**6**].

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A New Proof of Euler's Inequality

So far the well known Euler's inequality $2r \le R$ connecting the inradius r and circumradius R of a triangle has seen many proofs. We present what we believe to be a new one.

Let a, b, c denote the side lengths, s the semiperimeter, r_a, r_b, r_c the exradii, (ABC) the area of a triangle ABC. $(ABC) = \sqrt{s(s-a)(s-b)(s-c)} = s\,r = (a\,b\,c)/(4\,R) = r_x\,(s-x)$, $x \in \{a, b, c\}$. Then,

$$\begin{aligned} 4\,r\,r_a &= 4\,\frac{(ABC)^2}{s\,(s-a)} = 4\,(s-b)\,(s-c) = (a+c-b)\,(a+b-c) = \\ &= a^2 - (b-c)^2 \le a^2, \end{aligned} \tag{9}$$

and by similar reasoning,

$$4\,r\,r_b \le b^2 \text{ and } 4\,r\,r_c \le c^2. \tag{10}$$

Multiplying inequalities (1,2) by parts we take

$$\begin{aligned} 64\,r^3\,r_a\,r_b\,r_c &\le a^2\,b^2\,c^2 \Leftrightarrow \\ 64\,r^3\,(ABC)^3 &\le 16\,R^2\,(ABC)^2\,(s-a)\,(s-b)\,(s-c) \Leftrightarrow \\ 4\,r^4\,s &\le R^2\,(s-a)\,(s-b)\,(s-c) \Leftrightarrow \\ 4\,r^4\,s^2 &\le R^2\,(ABC)^2 = R^2\,s^2\,r^2 \Leftrightarrow \\ 4\,r^2 &\le R^2, \end{aligned}$$

and Euler's inequality follows.

—Submitted by Elias Lampakis

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