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A negative answer to Ulam's Problem 19 from the Scottish Book

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To my family

Abstract

We give a negative answer to Ulam's Problem 19 from the Scottish Book asking *is a solid of uniform density which will float in water in every position a sphere*? Assuming that the density of water is 1, we show that there exists a strictly convex body of revolution $K \subset \mathbb{R}^3$ of uniform density $\frac{1}{2}$, which is not a Euclidean ball, yet floats in equilibrium in every orientation. We prove an analogous result in all dimensions $d \geq 3$.

1. Introduction

The following intriguing problem was proposed by Ulam [Ula60, Prob. 19]: *If a convex body $K \subset \mathbb{R}^3$ made of material of uniform density $\mathcal{D} \in (0,1)$ floats in equilibrium in any orientation* (*in water*, *of density* 1), *must K be a Euclidean ball*?

Schneider [Sch71] and Falconer [Fal83] showed that this is true, provided K is centrally symmetric and $\mathcal{D} = \frac{1}{2}$. No results are known for other densities $\mathcal{D} \in (0,1)$ and no counterexamples have been found so far.

The "two-dimensional version" of the problem is also very interesting. In this case, we consider floating logs of uniform cross-section, and seek the ones that will float in every orientation with the axis horizontal. If $\mathcal{D} = \frac{1}{2}$, Auerbach [Aue38] has exhibited logs with non-circular cross-section, both convex and non-convex, whose boundaries are so-called Zindler curves [Zin21]. More recently, Bracho, Montejano and Oliveros [BMO04] showed that for densities $\mathcal{D}$ corresponding to perimetral densities $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{2}{5}$, the answer is affirmative,

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while Wegner proved that for some other values of $\mathcal{D} \neq \frac{1}{2}$ the answer is negative [Weg03], [Weg07]; see also related results of Várkonyi [Vár13], [Vár09]. Overall, the case of general $\mathcal{D} \in (0,1)$ is notably involved and widely open.

In this paper we prove the following result.

THEOREM 1. *Let $d \geq 3$. There exists a strictly convex* non-centrally-symmetric *body of revolution $K \subset \mathbb{R}^d$ that floats in equilibrium in every orientation at the level $\frac{\mathrm{vol}_d(K)}{2}$.*

This gives

THEOREM 2. *The answer to Ulam's Problem* 19 *is negative*; *i.e.*, *there exists a convex body $K \subset \mathbb{R}^3$ of density $\mathcal{D} = \frac{1}{2}$ that is not a Euclidean ball*, *yet floats in equilibrium in every orientation.*

Our bodies will be *small perturbations of the Euclidean ball.* We combine our recent results from [Rya20] together with work of Olovjanischnikoff [Olo41] and then use the machinery developed together with Nazarov and Zvavitch in [NRZ14]. The proofs of Theorem 1 for even and odd d are different. For even d, we solve a finite moment problem to obtain our body as a *local* perturbation of the Euclidean ball. The case of odd d is more involved. To control the perturbation, we use the properties of the spherical Radon transform [Gar06, pp. 427–436], [Hel99, Ch. III, pp. 93–99].

We refer the reader to [CFG91, pp. 19–20], [Gar06, pp. 376–377], [Gil91], [Mau15, pp. 90–93] and [Ula60] for an exposition of known results related to the problem.

This paper is structured as follows. In Section 2, we recall all the necessary notions and statements needed to prove the main result. In Section 3, we reduce the problem to finding a non-trivial solution to a system of two integral equations. In Section 4, we prove Theorem 1 for even d. In Section 5, we give the proof of Theorem 1 for odd d and prove Theorem 2. In Appendix A, we present the proof of Theorem 3 given in [Olo41]. We prove the converse part of Theorem 4 in Appendix B.

2. Notation and auxiliary results

Let $\mathbb{N} = \{1, 2, \dots,\}$ be the set of natural numbers. A convex body $K \subset \mathbb{R}^d$, $d \geq 2$, is a convex compact set with non-empty interior $\mathrm{int}\, K$. The boundary of K is denoted by ∂K. We say that K is strictly convex if ∂K does not contain a segment. We say that K is origin symmetric if $K = -K$ and centrally symmetric if there exists $p \in \mathbb{R}^d$ such that $K - p = \{q - p : q \in K\}$ is origin symmetric. Let $S^{d-1} = \{\xi \in \mathbb{R}^d : \sum_{j=1}^d \xi_j^2 = 1\}$ be the unit sphere in $\mathbb{R}^d$ centered at the origin, and let $B_2^d = \{p \in \mathbb{R}^d : \sum_{j=1}^d p_j^2 \leq 1\}$ be the Euclidean unit ball centered at the origin. We let $e_1, \dots, e_d$ be the standard basis in $\mathbb{R}^d$. Given $\xi \in S^{d-1}$, we denote by $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ the subspace orthogonal to ξ, where $p \cdot \xi = p_1\xi_1 + \cdots + p_d\xi_d$ is the usual inner product in $\mathbb{R}^d$. For $p \in \mathbb{R}^d$ we put $|p| = \sqrt{p_1^2 + \cdots + p_d^2}$. We also denote by $\mathcal{B}(\xi, \rho) = \{p \in S^{d-1} : p \cdot \xi > \rho\}$ the spherical cap centered at $\xi \in S^{d-1}$ of radius $\rho \in [-1, 1)$; we tacitly assume that $\mathcal{B}(\xi, -1) = S^{d-1}$. We will use the notation $\mathrm{vol}_d(U)$ for the d-dimensional Lebesgue measure of a Lebesgue-measurable set $U \subset \mathbb{R}^d$. We follow [Sch14] by writing $\kappa_d = \mathrm{vol}_d(B_2^d)$. Let W_j be a j-dimensional plane in $\mathbb{R}^d$, $1 \le j \le d$. We say that a plane W_j is the supporting plane of a convex body K if $K \cap W_j \neq \emptyset$, but $\mathrm{int}\, K \cap W_j = \emptyset$. The *center of mass* of a compact convex set $K \subset W_j$ with a non-empty relative interior will be denoted by $\mathcal{C}(K) = \frac{1}{\mathrm{vol}_j(K)} \int_K p\, dp$, where dp stands for integration with respect the usual Lebesgue measure. In fact, the notation dp (or dw, etc.) will always mean $d\mathscr{H}_j(p)$, $d\mathscr{H}_j(w)$, where $\mathscr{H}_j$ is j-dimensional Hausdorff measure in $\mathbb{R}^j$ or S^{j-1}, for the appropriate $j = 1, \ldots, d$. Given two sets A and B in $\mathbb{R}^d$, we denote by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ their symmetric difference, and by $A + B = \{a + b \in \mathbb{R}^d : a \in A, b \in B\}$ their Minkowski sum. We will use the notation $S(K)$ for the surface area of a convex body $K \subset \mathbb{R}^d$, which is defined via the Minkowski content $S(K) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}_d(K + \varepsilon B_2^d) - \mathrm{vol}_d(K)}{\varepsilon}$. Let $k \in \mathbb{N}$. We say that a function $h : \mathbb{R} \to \mathbb{R}$ supported on a closed interval $[a, b] \subset \mathbb{R}$, $a < b$, is in C^k (in C^∞) if it has continuous derivatives up to order k (of all orders). We define its norm as $\|h\|_{C^k} = \sum_{m=0}^k \max_{\{s \in [a,b]\}} |h^{(m)}(s)|$, where $h^{(m)}$ is the mth derivative of h. We say that a convex body $K \subset \mathbb{R}^d$ is of class C^k if K has a C^k-smooth boundary, i.e., for every point $z \in \partial K$, there exists a neighborhood U_z of z in $\mathbb{R}^d$ such that $\partial K \cap U_z$ can be written as a graph of a function having continuous partial derivatives up to the kth order.

Let $d \ge 3$, let $K \subset \mathbb{R}^d$ be a convex body, and let $\delta \in (0, \mathrm{vol}_d(K))$ be fixed. Given a direction $\xi \in S^{d-1}$ and $t = t(\xi) \in \mathbb{R}$, we call a hyperplane

$$H(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi = t\}$$

the *cutting hyperplane* of K in the direction ξ if it cuts out of K the given volume δ, i.e., if

$$\mathrm{vol}_d(K \cap H^-(\xi)) = \delta, \qquad \text{where} \quad H^-(\xi) = \{p \in \mathbb{R}^d : p \cdot \xi \le t(\xi)\} \tag{1}$$

(see Figure 1).

We recall several well-known facts and definitions; see [dLVP25, Ch. XXIV], [Lew88, Ch. 2], [Tup13, Ch. 4], [Zhu36, Hydrostatics, Part I].

Definition 1. Let $\xi \in S^{d-1}$, and let $\mathcal{C}_\delta(\xi)$ be the center of mass of the submerged part $K \cap H^-(\xi)$ satisfying (1). We say that K *floats in equilibrium in the direction* ξ *at the level* δ if the line $\ell(\xi)$ passing through $\mathcal{C}(K)$ and $\mathcal{C}_\delta(\xi)$ is orthogonal to the "free water surface" $H(\xi)$; i.e., the line $\ell(\xi)$ is "vertical"

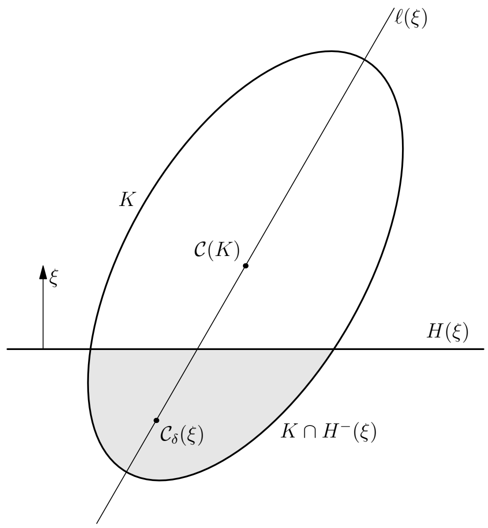

Figure 1. A body K, its submerged part $K \cap H^-(\xi)$ and the line $\ell(\xi)$ passing through $\mathcal{C}(K)$ and $\mathcal{C}_\delta(\xi)$.

(parallel to ξ — see Figure 1). We say that K *floats in equilibrium in every orientation at the level* δ if $\ell(\xi)$ is parallel to ξ for every $\xi \in S^{d-1}$.

Definition 2. The geometric locus $\{\mathcal{C}_\delta(\xi) : \xi \in S^{d-1}\}$ is called the *surface of centers* $\mathcal{S} = \mathcal{S}_\delta$ or the *surface of buoyancy*.

It is well known that $\mathcal{S}$ encloses a strictly convex body $L(\mathcal{S})$ (see Theorem 5 in Appendix B). We will use the notation $\mathcal{C}(\mathcal{S})$ for the center of mass of $L(\mathcal{S})$.

Now we recall the notion of *characteristic points* of a family of hyperplanes (cf. [BG92, pp. 107–110], [Wea55, pp. 48–50], or [Zal75, pp. 26–54]).

Definition 3. Let $d \geq 2$, let $\xi_0 \in S^{d-1}$, and let $\rho \in [-1, 1)$. Consider a family $\mathcal{Q}$ of hyperplanes in $\mathbb{R}^d$ such that for every direction $\xi \in \mathcal{B}(\xi_0, \rho)$, there exists a hyperplane in $\mathcal{Q}$ orthogonal to ξ. Assume also that for any $H \in \mathcal{Q}$, for any $(d-2)$-dimensional subspace Γ parallel to H and for any sequence $\{H_k\}_{k=1}^{\infty}$ of hyperplanes $H_k \in \mathcal{Q}$ converging to H as $k \to \infty$ and parallel to Γ, the limit $\Pi_\Gamma(H) = \lim_{k\to\infty} H \cap H_k$ exists. Given $H \in \mathcal{Q}$, we call a point $e \in H$ *the characteristic point of* $\mathcal{Q}$ *with respect to* H if for any Γ and $\{H_k\}_{k=1}^{\infty}$, as above, we have $e \in \Pi_\Gamma(H)$.

We remark that the characteristic point, if it exists, is unique. For, it must belong to $\bigcap \Pi_\Gamma(H)$, where the intersection is taken over all Γ parallel to H and the latter set is a singleton or empty.

We will need the following result from [Olo41]. (See the lemma on pages 114–117 and Remark 1 on p. 117.)

THEOREM 3. *Let $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body, and let $\delta \in (0, \mathrm{vol}_d(K))$. The characteristic points of the family of cutting hyperplanes $\{H(\xi) : \xi \in S^{d-1}\}$ for which (1) holds are the centers of mass of the sections $\{K \cap H(\xi) : \xi \in S^{d-1}\}$.*

Conversely, if the characteristic points of the family of hyperplanes $\{H(\xi) : \xi \in S^{d-1}\}$ intersecting the interior of K and corresponding to the sections $\{K \cap H(\xi) : \xi \in S^{d-1}\}$ coincide with the centers of mass of these sections, then the function $\xi \mapsto \mathrm{vol}_d(K \cap H^-(\xi))$ is constant on S^{d-1} and the constant is equal to some $\delta \in (0, \mathrm{vol}_d(K))$.

Since the reference [Olo41] is not readily available, for the reader's convenience we present the proof of Theorem 3 in Appendix A.

To define the *moments of inertia* (see [Zhu36, p. 553]), consider a convex body K and a hyperplane $H(\xi)$ for which (1) holds. Choose any $(d-2)$-dimensional plane $\Pi \subset H(\xi)$ passing through the center of mass $\mathcal{C}(K \cap H(\xi))$ and let $\eta_1, \dots, \eta_{d-2}, \eta_{d-1}$ be an orthonormal basis of $\xi^\perp = \{p \in \mathbb{R}^d : p \cdot \xi = 0\}$ such that

$$\Pi = \mathcal{C}(K \cap H(\xi)) + \mathrm{span}(\eta_1, \dots, \eta_{d-2}), \quad H(\xi) = \mathcal{C}(K \cap H(\xi)) + \xi^\perp. \tag{2}$$

Definition 4. The moment of inertia $I_{K \cap H(\xi)}(\Pi)$ of $K \cap H(\xi)$ with respect to Π is calculated by summing $\mathrm{dist}(\Pi, p)^2$ for every "particle" p in the set $K \cap H(\xi)$ (see Figure 2), i.e.,

$$I_{K \cap H(\xi)}(\Pi) = \int\limits_{K \cap H(\xi)} \mathrm{dist}(\Pi, p)^2 \, dp = \int\limits_{K \cap H(\xi) - \mathcal{C}(K \cap H(\xi))} (q \cdot \eta_{d-1})^2 \, dq, \tag{3}$$

where $\mathrm{dist}(\Pi, p) = \min_{\{q \in \Pi\}} |p - q|$.

We will use the converse part of the following theorem; see [Rya20, Th. 1] or [FSWZ20, Th. 1.1].[1]

THEOREM 4. *Let $d \geq 3$, let $K \subset \mathbb{R}^d$ be a convex body and let $\delta \in (0, \mathrm{vol}_d(K))$.*

If K floats in equilibrium at the level δ in every orientation, then for all $\xi \in S^{d-1}$ and for all $(d-2)$-dimensional planes $\Pi \subset H(\xi)$ passing through the center of mass $\mathcal{C}(K \cap H(\xi))$, the cutting sections $K \cap H(\xi)$ have equal moments of inertia independent of ξ and Π.

[1] It is assumed in [FSWZ20] that in the case $\delta = \frac{\mathrm{vol}_d(K)}{2}$ the set of characteristic points of the cutting hyperplanes is a single point.

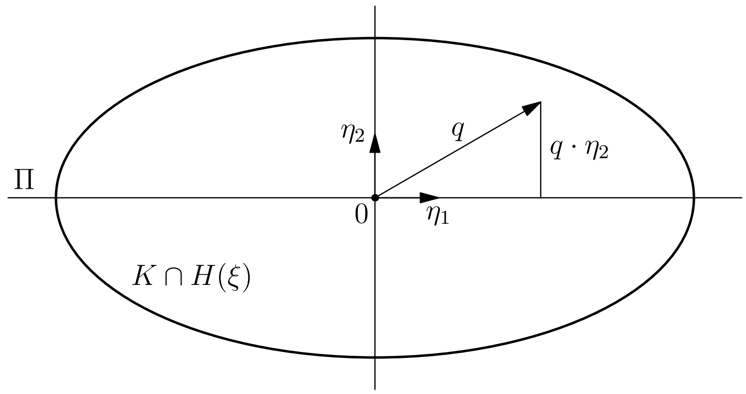

Figure 2. Two-dimensional body $K \cap H(\xi)$ with center of mass at the origin, and a line Π parallel to η_1; we have $\operatorname{dist}(\Pi, q)^2 = |q|^2 - (q \cdot \eta_1)^2 = (q \cdot \eta_2)^2$.

Conversely, let $\mathcal{S}$ be the surface of centers (see Definition 2), and let $\mathcal{C}(\mathcal{S}) = \mathcal{C}(K)$. If for all cutting hyperplanes $H(\xi)$, $\xi \in S^{d-1}$, and for all $(d-2)$-dimensional planes $\Pi \subset H(\xi)$ passing through the center of mass $\mathcal{C}(K \cap H(\xi))$, the cutting sections $K \cap H(\xi)$ have equal moments of inertia independent of ξ and Π, then K floats in equilibrium in every orientation at the level δ.

For the reader's convenience, we prove the converse part of this theorem in Appendix B[2].

Remark 1. Let $\delta = \frac{\mathrm{vol}_d(K)}{2}$. Since for any $\xi \in S^{d-1}$, $\mathcal{C}(K)$ is the arithmetic average of $\mathcal{C}(K \cap H^+(\xi))$ and $\mathcal{C}(K \cap H^-(\xi))$, the condition $\mathcal{C}(\mathcal{S}) = \mathcal{C}(K)$ is satisfied and $\mathcal{S}$ is symmetric with respect to $\mathcal{C}(K)$.

3. Reduction to a system of integral equations

Let $d \geq 3$. We follow the notation from [NRZ14]. We will be dealing with bodies of revolution

$$K_f = \{x \in \mathbb{R}^d : x_2^2 + x_3^2 + \cdots + x_d^2 \leq f(x_1)^2\}$$

obtained by the rotation of a smooth concave function supported on $[-R_1, R_2]$ about the x_1-axis. Let $L(s,t) = L_s(t) = st + h(s)$ be a linear function with slope $s \in \mathbb{R}$, and let

$$H(L_s) = \{x \in \mathbb{R}^d : x_d = L_s(x_1)\}$$

[2]It is assumed in [Rya20] that K is of class C^1. We give a slightly different proof that does not use this assumption.

be the corresponding hyperplane. The function h will be chosen later. For now it is enough to assume that it is infinitely smooth, not identically zero, supported on $[1-2\tau, 1-\tau]$ for some small $\tau > 0$, and h and sufficiently many of its derivatives are small. Let $-x = -x(s)$ and $y = y(s)$ be the first coordinates of the points of intersection of $\pm f$ and L (see Figure 3).

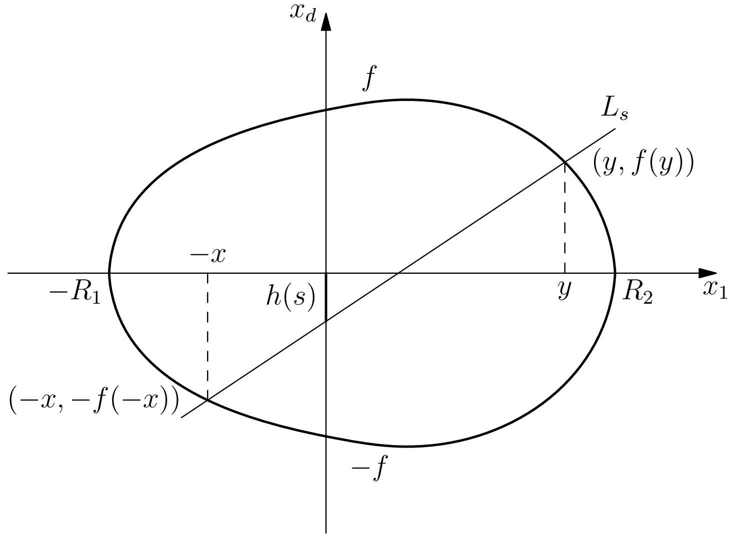

Figure 3. Sections of K_f and $H(L)$ by the (x_1, x_d)-plane.

To construct a system of two integral equations we will prove four lemmas. Consider the family of hyperplanes

$$\mathcal{F} = \{H(L_s) : s \in [0, \infty)\}. \tag{4}$$

LEMMA 1. *Let E be the set of characteristic points of $\mathcal{F}$. Then,*

$$E = \{(-h'(s), 0, \dots, 0, L(s, -h'(s))) \in \mathbb{R}^d : s \in [0, \infty)\}. \tag{5}$$

Proof. Let $\mathcal{G}$ be the family of lines $\mathcal{G} = \{\ell_s : s \in [0, \infty)\}$, where each line ℓ_s is the intersection of $H(L_s)$ and the $x_1 x_d$-plane. It is enough to show that

$$E \cap \{x_1 x_d\text{-plane}\} = \{(-h'(s), -sh'(s) + h(s)) \in \mathbb{R}^2 : s \in [0, \infty)\}.$$

We will use Definition 3. Let $s \in (1 - 2\tau, 1 - \tau)$, and let $\ell_s \in \mathcal{G}$. Choose any sequence $\{\ell_{s_k}\}_{k=1}^{\infty}$, $\ell_{s_k} \in \mathcal{G}$, converging to ℓ_s as $k \to \infty$, and let $\{u_{s_k}\}_{k=1}^{\infty}$, $\{u_{s_k}\} = \ell_s \cap \ell_{s_k}$, be the corresponding sequence of points of intersection. Solving the system of two linear equations we see that

$$u_{s_k} = \Big(\frac{h(s) - h(s_k)}{s_k - s}, s_k \frac{h(s) - h(s_k)}{s_k - s} + h(s_k)\Big).$$

Hence, $\lim_{s_k\to s} u_{s_k}$ exists, and the point $\lim_{s_k\to s} u_{s_k} = (-h'(s), -sh(s)+h(s))$ is the characteristic point of $\mathcal{G}$ with respect to ℓ_s.

Next, we observe that $(0,0)$ is the characteristic point of $\mathcal{G}$ with respect to $\ell_{1-2\tau}$. Indeed, it is enough to choose two sequences of lines in $\mathcal{G}$, $\{\ell_{s_k}\}_{k=1}^{\infty}$, $\{\ell_{s'_k}\}_{k=1}^{\infty}$, both converging to $\ell_{1-2\tau}$, such that $s_k \in (1-2\tau, 1-\tau)$ and $s'_k \in (0, 1-2\tau)$, and to use the fact that $\ell_{s'_k} \cap \ell_{1-2\tau} = \{(0,0)\}$ for any line $\ell_{s'_k}$ with $s'_k \in (0, 1-2\tau)$. Similarly, to show that $(0,0)$ is the characteristic point of $\mathcal{G}$ with respect to $\ell_{1-\tau}$, it is enough to choose the corresponding sequences $\{\ell_{s_k}\}_{k=1}^{\infty}$, $\{\ell_{s''_k}\}_{k=1}^{\infty}$, both converging to $\ell_{1-\tau}$, where $s_k \in (1-2\tau, 1-\tau)$ and $s''_k \in (1-\tau, \infty)$.

To finish the proof, it remains to observe that since h is supported by $[1-2\tau, 1-\tau]$, any two lines ℓ_s, $\ell_{s'}$, $s, s' \in [0, 1-2\tau) \cup (1-\tau, \infty)$, intersect at $(0,0)$. Hence, $(0,0)$ is the characteristic point of $\mathcal{G}$ with respect to any line ℓ_s for $s \in [0, 1-2\tau] \cup [1-\tau, \infty)$. $\square$

LEMMA 2. *Let $s > 0$. The condition*

$$\mathcal{C}(K_f \cap H(L_s)) = (-h'(s), 0, \dots, 0, L(s, -h'(s))) \tag{6}$$

reads as

$$\int\limits_{-x(s)}^{y(s)} (t + h'(s))(f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}}\, dt = 0. \tag{7}$$

Let

$$\Pi_1 = \{x \in H(L_s) : x_1 = -h'(s)\}, \qquad \Pi_j = \{x \in H(L_s) : x_j = 0\},$$

$j = 2, \dots, d-1$. *The moments of inertia conditions*

$$I_j = I_{K_f \cap H(L_s)}(\Pi_j) = \text{const}, \quad j = 1, \dots, d-1,$$

read as

$$I_1 = \kappa_{d-2}(1+s^2)^{\frac{3}{2}} \int\limits_{-x(s)}^{y(s)} (t + h'(s))^2 (f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}}\, dt = \text{const}, \tag{8}$$

$$I_j = \gamma_{d-2}\sqrt{1+s^2} \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d}{2}}\, dt = \text{const}, \tag{9}$$

where

$$\gamma_{d-2} = \int\limits_{B_2^{d-2}} p_j^2\, dp, \quad j = 2, \dots, d-1.$$

Proof. Fix $s > 0$. Observe that the slice $K_f \cap H(L_s) \cap H_t$ of the cutting section $K_f \cap H(L_s)$ by the hyperplane $H_t = \{x \in \mathbb{R}^d : x_1 = t\}$, $-x(s) < t < y(s)$, is the $(d-2)$-dimensional Euclidean ball

$$B_2^{d-2}((t,0,\dots,0,L(s,t)),r) = \{(t,x_2,\dots,x_{d-1},L(s,t)) : x_2^2+\cdots+x_{d-1}^2 \le r^2\}$$

of radius $r = \sqrt{f^2(t)-L^2(s,t)}$ centered at $(t,0,\dots,0,L(s,t))$. Hence, for the first coordinate of the center of mass in (6), we have

$$\begin{aligned}(10)\qquad &\int\limits_{-x(s)}^{y(s)} (t+h'(s))\,dt \int\limits_{B_2^{d-2}((t,0,\dots,0,L(s,t)),r)} dp \\ &\quad = \kappa_{d-2} \int\limits_{-x(s)}^{y(s)} (t+h'(s))(f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}}\,dt = 0.\end{aligned}$$

This gives (7).

Similarly, since the distance in $K_f \cap H(L_s)$ between the points

$$(t, x_2, \dots, x_d) \in K_f \cap H(L_s) \cap H_t$$

and

$$(-h'(s), x_2, \dots, x_d) \in K_f \cap H(L_s) \cap H_{-h'(s)}$$

is $\sqrt{1+s^2}|t+h'(s)|$, we have

$$\begin{aligned} I_1 &= \sqrt{1+s^2} \int\limits_{-x(s)}^{y(s)} (\sqrt{1+s^2}(t+h'(s))^2\,dt \int\limits_{B_2^{d-2}((t,0,\dots,0,L(s,t)),r)} dp \\ &= \kappa_{d-2}(1+s^2)^{\frac{3}{2}} \int\limits_{-x(s)}^{y(s)} (t+h'(s))^2 (f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}}\,dt,\end{aligned}$$

proving (8). Finally, the expression in the left-hand side of (9) for the other moments can be obtained as

$$\begin{aligned} I_j &= \sqrt{1+s^2} \int\limits_{-x(s)}^{y(s)} dt \int\limits_{B_2^{d-2}((t,0,\dots,0,L(s,t)),r)} p_j^2\,dp \\ &= \sqrt{1+s^2}\,\gamma_{d-2} \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d}{2}}\,dt. \qquad \square\end{aligned}$$

LEMMA 3. *Let $s_o \ge 0$, let K_f be as above, and let $\mathcal{F}$ be the family of hyperplanes defined as in (4) for $s \ge s_0$, so that (6) holds for $s \ge s_o$. Then for all $s > s_o$ and for all $(d-2)$-dimensional planes $\Pi \subset H(L_s)$ passing through* *the center of mass $\mathcal{C}(K_f\cap H(L_s))$, the cutting sections $K_f\cap H(L_s)$ have equal moments of inertia $I_{K_f\cap H(L_s)}(\Pi)$ independent of s and Π, provided (8) and (9) hold with the same constant on the right-hand side, which is independent of s and $j=1,\dots,d-1$.*

Proof. Let $s_o\ge 0$ and let $s>s_o$ be fixed. If $\Pi\subset H(L_s)$ is any $(d-2)$-dimensional plane passing through the center of mass $\mathcal{C}_s=\mathcal{C}(K_f\cap H(L_s))$, then by (3) we have

$$I_{K_f\cap H(L_s)}(\Pi)=\int\limits_{K_f\cap H(L_s)}((u-\mathcal{C}_s)\cdot\eta)^2\,du,$$

where $\eta=\eta_{d-1}$ is a unit vector in the hyperplane $H(L_s)-\mathcal{C}_s$ that is orthogonal to Π.

Let $\iota_1,\dots\iota_{d-1}$ be the orthonormal basis in $H(L_s)-\mathcal{C}_s$ such that $\iota_1\in\operatorname{span}\{e_1,e_d\}$ and $\iota_j=e_j$ for $j=2,\dots,d-1$. Decomposing η in this basis as $\sum_{j=1}^{d-1}\eta_{(j)}\iota_j$, we have

$$\begin{aligned}I_{K_f\cap H(L_s)}(\Pi)=&\sum_{j=1}^{d-1}\eta^2_{(j)}\int\limits_{K_f\cap H(L_s)}((u-\mathcal{C}_s)\cdot\iota_j)^2\,du\\&+\sum_{\substack{j,l=1\\ j\neq l}}^{d-1}\eta_{(j)}\eta_{(l)}\int\limits_{K_f\cap H(L_s)}((u-\mathcal{C}_s)\cdot\iota_j)((u-\mathcal{C}_s)\cdot\iota_l)\,du=J_1+J_2.\end{aligned}$$

Using the fact that η is a unit vector, together with (8) and (9), we have that J_1 is constant.

We claim that $J_2=0$. Indeed, if j is equal to 1, then arguing as in the previous lemma, and using the fact that $\int\limits_{B_2^{d-2}}p_l\,dp=0$ for $l=2,\dots,d-1$, we see that

$$\begin{aligned}&\int\limits_{K_f\cap H(L_s)}((u-\mathcal{C}_s)\cdot\iota_1)((u-\mathcal{C}_s)\cdot\iota_l)\,du\\&\qquad=\sqrt{1+s^2}\int\limits_{-x(s)}^{y(s)}(t+h'(s))\,dt\int\limits_{B_2^{d-2}((t,0,\dots,0,L(s,t)),r)}p_l\,dp=0.\end{aligned}$$

The case when $l=1$ is similar.

If $j \neq 1$, $l \neq 1$, then we use the fact that $\int\limits_{B_2^{d-2}} p_j p_l \, dp = 0$ for $j, l = 2, \dots, d-1$, $j \neq l$, to obtain

$$\int\limits_{K_f \cap H(L_s)} ((u - \mathcal{C}_s) \cdot \iota_j)((u - \mathcal{C}_s) \cdot \iota_l) \, du = \int\limits_{-x(s)}^{y(s)} dt \int\limits_{B_2^{d-2}((t,0,\dots,0,L(s,t)),r)} p_j p_l \, dp = 0.$$

Therefore, $I_{K_f \cap H(L_s)}(\Pi)$ is a constant independent of s and of the arbitrarily chosen Π. The lemma is proved. $\square$

Lemma 4. *Let $s_o \geq 0$. Assume that (7) is valid for all $s > s_o$. Then (9) holds for all $s > s_o$ with the constant independent of s if and only if (8) holds for all $s > s_o$ with the constant independent of s.*

Proof. We recall that

$$L(s,t) = st + h(s), \quad f(y(s)) = L(s, y(s)), \quad f(-x(s)) = L(s, -x(s)) \tag{11}$$

for $s \in \mathbb{R}$. Let $s_o \geq 0$, and we let $s > s_o$. We rewrite (9) as

$$\int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d}{2}} \, dt = \frac{\text{const}}{\gamma_{d-2} \sqrt{1 + s^2}}$$

and differentiate both sides with respect to s using (11). We have

$$\int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}} (st + h(s))(t + h'(s)) \, dt = \frac{\text{const}\, s}{d\gamma_{d-2} \, (1 + s^2)^{\frac{3}{2}}}.$$

Adding and subtracting $sh'(s)$ in the middle parentheses under the integral and using (7), the last equality yields

$$s \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}} (t + h'(s))^2 \, dt = \frac{\text{const}\, s}{d\gamma_{d-2} \, (1 + s^2)^{\frac{3}{2}}}.$$

Canceling s and passing to polar coordinates,

$$d\gamma_{d-2} = \frac{d}{d-2} \int\limits_{B_2^{d-2}} |p|^2 \, dp = \frac{d}{d-2} \int\limits_{S^{d-3}} d\omega \int\limits_0^1 r^{2+d-3} \, dr = \frac{\omega(S^{d-3})}{d-2} = \kappa_{d-2},$$

where $\omega(S^{d-3})$ is the surface area of S^{d-3}, we have (8).

Now we prove the converse statement. Fix any $j = 2, \dots, d-1$. We rewrite the first equality in (9) as

$$\frac{I_j(s)}{\gamma_{d-2}\sqrt{1+s^2}} = \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d}{2}}\, dt$$

and differentiate both sides with respect to s. Using (7) and (8), we see that

$$\left(\frac{I_j(s)}{\sqrt{1+s^2}}\right)' = \frac{I_j'(s)(1+s^2) - sI_j(s)}{(1+s^2)^{\frac{3}{2}}} = -\frac{\mathrm{const}\, s}{(1+s^2)^{\frac{3}{2}}}, \tag{12}$$

where the second equality above is obtained follows. Using (11) we differentiate the first equality in (9) to obtain

$$I_j'(s)(1+s^2) = \gamma_{d-2}\, s\sqrt{1+s^2} \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d}{2}}\, dt$$

$$- d\gamma_{d-2}(1+s^2)^{\frac{3}{2}} \int\limits_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{\frac{d-2}{2}}(st + h(s))(t + h'(s))\, dt.$$

Adding and subtracting $sh'(s)$ in the middle parentheses under the second integral and using (7), the fact that $d\gamma_{d-2} = \kappa_{d-2}$ and the second equality in (8), we have

$$I_j'(s)(1+s^2) - sI_j(s) = sI_j(s) - sI_1 - sI_j(s) = -sI_1 = -\mathrm{const}\, s.$$

This gives the second equality in (12), i.e.,

$$I_j'(s) - \frac{s}{1+s^2} I_j(s) + \mathrm{const}\,\frac{s}{1+s^2} = 0.$$

Solving this linear ODE with an integrating factor $\frac{1}{\sqrt{1+s^2}}$, we have

$$I_j(s) = \sqrt{1+s^2}\left(\frac{\mathrm{const}}{\sqrt{1+s^2}} + c_1\right) = \mathrm{const} + c_1\sqrt{1+s^2}$$

with some constant c_1. Since I_j is bounded on $[s_o, \infty)$, $c_1 = 0$, and we obtain the converse part of the lemma. □

Let

$$f_o(t) = \sqrt{1-t^2}, \qquad L_o(s,t) = st, \qquad x_o(s) = y_o(s) = \frac{1}{\sqrt{1+s^2}}, \tag{13}$$

where f_o describes the boundary of the unit Euclidean ball, L_o corresponds to the linear subspace passing through the origin with $h \equiv 0$, and x_o, y_o are the first coordinates of the points of intersection of $\pm f$ and L_o. Our goal is to prove the following proposition.

PROPOSITION 1. *Let $n = \frac{d}{2}$. A body K_f floats in equilibrium in every orientation at the level $\frac{\mathrm{vol}_d(K)}{2}$, provided for all $s > 0$,*

$$\int_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^n \, dt = \int_{-x_o(s)}^{y_o(s)} (f_o(t)^2 - L_o(s,t)^2)^n \, dt = \frac{\mathrm{const}}{\sqrt{1+s^2}}, \tag{14}$$

$$\int_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{n-1} \frac{\partial L(s,t)}{\partial s} \, dt = 0. \tag{15}$$

We remark that (14) and (15) are similar to equations (4) and (5) from [NRZ14].

Proof. Observe that $H(L_0)$ divides K_f into two parts of equal volume. Also, (15) is the same as (7) of Lemma 2. Thus, by Lemmas 1 and 2 the characteristic points of the family of hyperplanes $\{H(L_s),\ s \in [0,\infty)\}$, are exactly the centers of mass of the sections $K \cap H(L_s)$. Hence, we can apply the converse part of Theorem 3 to conclude that they are the cutting hyperplanes at the level $\frac{\mathrm{vol}_d(K)}{2}$.

On the other hand, observing that conditions (14), (15) are the same as (9) and (7), by Lemma 4, condition (8) also holds. Therefore, by Lemma 3, the cutting sections have equal moments of inertia for all $(d-2)$-dimensional planes passing through the centers of mass of these sections. By Remark 1, all conditions of the converse part of Theorem 4 are satisfied, and the proposition follows. $\square$

In order to construct a counterexample, we will choose the *perturbation function* h with the properties described at the beginning of this section. The convex body corresponding to any such function will automatically be asymmetric since not all its sections dividing the volume in half will pass through a single point.

4. The case of even $d \geq 4$

Note that in this case $n = \frac{d}{2} \in \mathbb{N}$. Our argument is very similar to the one in Section 3 of [NRZ14]. Our body K_f will be a *local* perturbation of the Euclidean ball; i.e., the resulting function $f(t)$ will be equal to $\sqrt{1-t^2}$ everywhere on $[-1,1]$ except $[-\frac{1}{\sqrt{1+(1-2\tau)^2}}, -\frac{1}{\sqrt{1+(1-\tau)^2}}] \cup [\frac{1}{\sqrt{1+(1-\tau)^2}}, \frac{1}{\sqrt{1+(1-2\tau)^2}}]$ for some small $\tau > 0$.

Equations (11) show that to define f, it is enough to define two decreasing functions $x(s)$, $y(s)$ on $[0,+\infty)$. Our functions $x(s)$ and $y(s)$ will coincide with x_o and y_o for all $s \notin [1-2\tau, 1-\tau]$, where x_o, y_o are defined by (13). Since the curvature of the semicircle is strictly positive, the resulting function f will be strictly concave if x and y are close to x_o and y_o in C^2.

We will make our construction in several steps. First, we *define* $x = x_o$, $y = y_o$ on $[1,\infty)$. Second, we will express equations (14), (15) purely in terms of x and y (see (18) and (19) below). Then we will use these new equations to extend the functions x and y to $[1-3\tau, 1]$. We will be able to do it if τ and h are sufficiently small. Moreover, the extensions will coincide with x_o and y_o on $[1-\tau, 1]$ and will be close to x_o and y_o up to two derivatives on $[1-3\tau, 1-\tau]$. Then, we will show that our extensions automatically coincide with x_o and y_o on $[1-3\tau, 1-2\tau]$ as well. This will allow us to put $x = x_o$, $y = y_o$ on the remaining interval $[0, 1-3\tau]$ and get a nice smooth function. Finally, we will show that equations (14), (15) will be satisfied up to $s = 0$, thus finishing the proof.

Step 1. We put $x = x_o$, $y = y_o$ on $[1,\infty)$.

Step 2. To construct x, y on $[1-3\tau, 1]$, we will make some technical preparations. First, we will differentiate equations (14), (15) a few times to obtain a system of *four* integral equations with *four* unknown functions x, y, x', y'. Next, we will apply Lemma 8 and Remark 2 from [NRZ14, pp. 63–66] to show that there exists a solution x, y, x', y' of the constructed system of integral equations on $[1-3\tau, 1]$, which coincides with x_o, y_o, $\frac{dx_o}{ds}$, $\frac{dy_o}{ds}$ on $[1-\tau, 1]$. Finally, we will prove that the x and y components of that solution give a solution of (14), (15) with f defined by (11).

Differentiating equation (14) $n+1$ times and equation (15) n times with respect to s and using (11), we obtain

$$(-2)^n n!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}\Big)^n \frac{dx}{ds}(s) + \Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}\Big)^n \frac{dy}{ds}(s)\Big] + \int_{-x(s)}^{y(s)} \Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big((f(t)^2 - L(s,t)^2)^n\Big)\,dt = \Big(\frac{d}{ds}\Big)^{n+1}\Big(\frac{\text{const}}{\sqrt{1+s^2}}\Big) \tag{16}$$

and

$$(-2)^{n-1}(n-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}\frac{dx}{ds}(s) + \Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}\frac{dy}{ds}(s)\Big] + \partial s(s,t)\Big)\,dt = 0. \tag{17}$$

When $s \le 1$, the integral term I in (16) can be split as

$$\begin{aligned} I &= \int_{-x(s)}^{y(s)} \Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big((f(t)^2 - L(s,t)^2)^n\Big)\,dt \\ &= \Big(\int_{-x(s)}^{-x_o(1)} + \int_{y_o(1)}^{y(s)}\Big)\Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big((f(t)^2 - L(s,t)^2)^n\Big)\,dt + \Xi_1(s), \end{aligned}$$

where

$$\Xi_1(s) = \int\limits_{-x_o(1)}^{y_o(1)} \left(\frac{\partial}{\partial s}\right)^{n+1} \Big((f_o(t)^2 - L(s,t)^2)^n\Big)\, dt.$$

Making the change of variables $t = -x(\sigma)$ in the integral $\int_{-x(s)}^{-x_o(1)}$ and $t = y(\sigma)$ in the integral $\int_{y_o(1)}^{y(s)}$ and using (11), we obtain

$$\begin{aligned} I = &-\int\limits_s^1 \left(\frac{\partial}{\partial s}\right)^{n+1} \Big(L(\sigma, -x(\sigma))^2 - L(s, -x(\sigma))^2\Big)^n \frac{dx}{ds}(\sigma)\, d\sigma \\ &-\int\limits_s^1 \left(\frac{\partial}{\partial s}\right)^{n+1} \Big(L(\sigma, y(\sigma))^2 - L(s, y(\sigma))^2\Big)^n \frac{dy}{ds}(\sigma)\, d\sigma \;+\; \Xi_1(s). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\int\limits_{-x(s)}^{y(s)} \left(\frac{\partial}{\partial s}\right)^n \Big((f(t)^2 - L(s,t)^2)^{n-1} \frac{\partial L}{\partial s}(s,t)\Big)\, dt \\ &\quad = -\int\limits_s^1 \left(\frac{\partial}{\partial s}\right)^n \Big(\Big(L(\sigma, -x(\sigma))^2 - L(s, -x(\sigma))^2\Big)^{n-1} \frac{\partial L}{\partial s}(s, -x(\sigma))\Big) \frac{dx}{ds}(\sigma)\, d\sigma \\ &\quad -\int\limits_s^1 \left(\frac{\partial}{\partial s}\right)^n \Big(\Big(L(\sigma, y(\sigma))^2 - L(s, y(\sigma))^2\Big)^{n-1} \frac{\partial L}{\partial s}(s, y(\sigma))\Big) \frac{dy}{ds}(\sigma)\, d\sigma \;+\; \Xi_2(s), \end{aligned}$$

where

$$\Xi_2(s) = \int\limits_{-x_o(1)}^{y_o(1)} \left(\frac{\partial}{\partial s}\right)^n \Big((f_o(t)^2 - L(s,t)^2)^{n-1} \frac{\partial L}{\partial s}(s,t)\Big)\, dt.$$

To reduce the resulting system of integro-differential equations to a pure system of integral equations we add two independent unknown functions x', y' and two new relations:

$$x(s) = -\int\limits_s^1 x'(\sigma)\, d\sigma + x_o(1), \qquad y(s) = -\int\limits_s^1 y'(\sigma)\, d\sigma + y_o(1).$$

We rewrite our equations (16), (17) as follows:

$$
\begin{aligned}
&(-2)^n n!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}\Big)^n x'(s) \;+\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}\Big)^n y'(s)\Big]\\
&\quad-\int\limits_s^1\Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big(L(\sigma,-x(\sigma))^2-L(s,-x(\sigma))^2\Big)^n x'(\sigma)\,d\sigma\\
&\quad-\int\limits_s^1\Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big(L(\sigma,y(\sigma))^2-L(s,y(\sigma))^2\Big)^n y'(\sigma)\,d\sigma\;+\Xi_1(s)\\
&=\Big(\frac{d}{ds}\Big)^{n+1}\Big(\frac{\text{const}}{\sqrt{1+s^2}}\Big)
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&(-2)^{n-1}(n-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}x'(s)\\
&\quad+\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}y'(s)\Big]\\
&\quad-\int\limits_s^1\Big(\frac{\partial}{\partial s}\Big)^n\Big(\Big(L(\sigma,-x(\sigma))^2-L(s,-x(\sigma))^2\Big)^{n-1}\frac{\partial L}{\partial s}(s,-x(\sigma))\Big)\,x'(\sigma)\,d\sigma\\
&\quad-\int\limits_s^1\Big(\frac{\partial}{\partial s}\Big)^n\Big(\Big(L(\sigma,y(\sigma))^2-L(s,y(\sigma))^2\Big)^{n-1}\frac{\partial L}{\partial s}(s,y(\sigma))\Big)\,y'(\sigma)\,d\sigma\\
&\quad+\Xi_2(s)=0.
\end{aligned}
\tag{19}
$$

Now we rewrite our system in the form

$$
\mathbf{G}(s,Z(s))=\int\limits_s^1\mathbf{\Theta}(s,\sigma,Z(\sigma))\,d\sigma+\mathbf{\Xi}(s). \tag{20}
$$

Here

$$
Z=\begin{pmatrix} x\\ y\\ x'\\ y'\end{pmatrix},
$$

$$\mathbf{G}(s,Z)=\begin{pmatrix} x \\ y \\ (-2)^n n!\Big[\Big(L\frac{\partial L}{\partial s}\Big|_{(s,-x)}\Big)^n x' + \Big(L\frac{\partial L}{\partial s}\Big|_{(s,y)}\Big)^n y'\Big] \\ (-2)^{n-1}(n-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x)} x' + \Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,y)} y'\Big] \end{pmatrix},$$

$$\mathbf{\Theta}(s,\sigma,Z) = -\begin{pmatrix} x' \\ y' \\ \Theta_1 \\ \Theta_2 \end{pmatrix},$$

where

$$\begin{aligned} \Theta_1 &= -\Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big(L(\sigma,-x)^2 - L(s,-x)^2\Big)^n x' \\ &\quad - \Big(\frac{\partial}{\partial s}\Big)^{n+1}\Big(L(\sigma,y)^2 - L(s,y)^2\Big)^n y', \\ \Theta_2 &= -\Big(\frac{\partial}{\partial s}\Big)^{n}\Big(\Big(L(\sigma,-x)^2 - L(s,-x)^2\Big)^{n-1}\frac{\partial L}{\partial s}(s,-x)\Big) x' \\ &\quad - \Big(\frac{\partial}{\partial s}\Big)^{n}\Big(\Big(L(\sigma,y)^2 - L(s,y)^2\Big)^{n-1}\frac{\partial L}{\partial s}(s,y)\Big) y', \end{aligned}$$

and

$$\mathbf{\Xi}(s) = \begin{pmatrix} x_o(1) \\ y_o(1) \\ -\Xi_1(s) + \Big(\frac{d}{ds}\Big)^{n+1}\Big(\frac{\mathrm{const}}{\sqrt{1+s^2}}\Big) \\ -\Xi_2(s) \end{pmatrix}.$$

Note that $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are well defined and infinitely smooth for all $s, \sigma \in (0,1]$ and $Z \in \mathbb{R}^4$. Observe also that

$$D_Z\mathbf{G}\Big|_{(s,Z)} = \begin{pmatrix} \mathbf{I} & 0 \\ * & \mathbf{A} \end{pmatrix},$$

where

$$\begin{aligned} \mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{A} = \mathbf{A}(s,x,y) \\ &= \begin{pmatrix} (-2)^n n!\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x)}\Big)^n & (-2)^n n!\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y)}\Big)^n \\ (-2)^{n-1}(n-1)!\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x)} & (-2)^{n-1}(n-1)!\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{n-1}\frac{\partial L}{\partial s}\Big)\Big|_{(s,y)} \end{pmatrix}. \end{aligned}$$

The function

$$Z_o(s) = \begin{pmatrix} x_o(s) \\ y_o(s) \\ \frac{dx_o}{ds}(s) \\ \frac{dy_o}{ds}(s) \end{pmatrix}$$

solves the system (20) with $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ corresponding to $h \equiv 0$ (we will denote them by $\mathbf{G}_o$, $\mathbf{\Theta}_o$, $\mathbf{\Xi}_o$) on $[\frac{1}{2}, 1]$.

We claim that

$$\det\left(D_Z \mathbf{G}_o\Big|_{(s, Z_o(s))}\right) = \det(\mathbf{A}_o(s, x_o(s), y_o(s))) \neq 0 \qquad \forall s \in (0, 1]. \tag{21}$$

Indeed, since the matrix $\mathbf{A}_o(s, x_o(s), y_o(s))$ is of the form

$$\begin{pmatrix} (-2)^n n!(sx_o(s))^n & (-2)^n n!(sy_o(s))^n \\ (-2)^{n-1}(n-1)!(sx_o(s))^{n-1}(-x_o(s)) & (-2)^{n-1}(n-1)!(sy_o(s))^{n-1} y_o(s) \end{pmatrix},$$

its sign pattern is

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix} \text{ when } n \text{ is even and } \begin{pmatrix} - & - \\ - & + \end{pmatrix} \text{ when } n \text{ is odd.}$$

Thus, (21) follows. In particular,

$$\det\left(D_Z \mathbf{G}_o\Big|_{(1, Z_o(1))}\right) \neq 0.$$

Lemma 8 from [NRZ14, p. 63] then implies that we can choose some small $\tau > 0$ and, for any fixed $k \in \mathbb{N}$, construct a solution $Z(s)$ of (20) that is C^k-close to $Z_o(s)$ on $[1 - 3\tau, 1]$, whenever $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are sufficiently close to $\mathbf{G}_o$, $\mathbf{\Theta}_o$, $\mathbf{\Xi}_o$ in C^k on certain compact sets. Since $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ and their derivatives are some explicit (integrals of) polynomials in Z, s, σ, $h(s)$, and the derivatives of $h(s)$, this closeness condition will hold if h and sufficiently many of its derivatives are close enough to zero. Moreover, since h vanishes on $[1 - \tau, 1]$, the assumptions of Remark 2 from [NRZ14, p. 66] are satisfied and we have $Z(s) = Z_o(s)$ on $[1 - \tau, 1]$.

To prove that the x and y components of the solution we found give a solution of (14), (15) with f defined by (11), we consider the functions

$$F(s) := \int_{-x(s)}^{y(s)} \left(f(s,t)^2 - L(s,t)^2\right)^n dt - \frac{\text{const}}{\sqrt{1 + s^2}},$$

$$H(s) := \int_{-x(s)}^{y(s)} \left(f(s,t)^2 - L(s,t)^2\right)^{n-1} \frac{\partial L}{\partial s}(s,t)\, dt.$$

Since equations (18) and (19) of our system (20) were obtained by the differentiation of equations (14), (15), we have

$$\left(\frac{d}{ds}\right)^{n+1} F(s) = 0, \qquad \left(\frac{d}{ds}\right)^{n} H(s) = 0$$

on $[1-3\tau, 1]$. Hence, F and H are polynomials on $[1-3\tau, 1]$. Since $h(s) = 0$, $x(s) = x_o(s)$, $y(s) = y_o(s)$ on $[1-\tau, 1]$, F and H vanish on $[1-\tau, 1]$ and, therefore, identically. Thus, we conclude that the x and y components of the solutions of (18), (19) solve (14), (15) on $(1-3\tau, 1]$. Step 2 is completed.

Step 3. We claim that $x = x_o$, $y = y_o$ on $[1-3\tau, 1-2\tau]$; i.e., the *perturbed solution* returns to the semicircle. Since h is supported on $[1-2\tau, 1-\tau]$, we have $L = L_o = st$ and $\frac{\partial}{\partial s}L(s,t) = t$ for $s \in [1-3\tau, 1-2\tau]$. It follows that every time we differentiate equation (14) (with respect to s) we can divide the result by s to obtain

$$\int_{-x(s)}^{y(s)} (f(t)^2 - L_o(s,t)^2)^{n-k} t^{2k} dt = \int_{-x_o(s)}^{y_o(s)} (f_o(t)^2 - L_o(s,t)^2)^{n-k} t^{2k} dt \tag{22}$$

for $k \le n$. If we take $k = n$ in (22), we get

$$\int_{-x(s)}^{y(s)} t^{2n}\, dt = \int_{-x_o(s)}^{y_o(s)} t^{2n}\, dt. \tag{23}$$

Similarly, for $k \le n-1$, equation (15) implies that

$$\begin{aligned} &\int_{-x(s)}^{y(s)} (f(t)^2 - L_o(s,t)^2)^{n-1-k} t^{2k+1}\, dt \\ &\quad = \int_{-x_o(s)}^{y_o(s)} (f_o(t)^2 - L_o(s,t)^2)^{n-1-k} t^{2k+1}\, dt = 0. \end{aligned} \tag{24}$$

Putting $k = n-1$ in (24), we get

$$\int_{-x(s)}^{y(s)} t^{2n-1}\, dt = 0 = \int_{-x_o(s)}^{y_o(s)} t^{2n-1}\, dt. \tag{25}$$

Equation (25) yields $x(s) = y(s)$, and the symmetry (with respect to 0) of the intervals $(-x_o(s), y_o(s))$, $(-x(s), y(s))$, together with (23), yield $(-x_o(s), y_o(s)) = (-x(s), y(s))$ for all $s \in [1-3\tau, 1-2\tau]$. Step 3 is completed.

Step 4. We put $x = x_o$, $y = y_o$ on $[0, 1-3\tau]$, which will result in a function f defined on $[-1, 1]$ and coinciding with $f_o(t) = \sqrt{1-t^2}$ outside small intervals around $\pm\frac{1}{\sqrt{2}}$. It remains to check that (14), (15) are valid for $s \in [0, 1-3\tau]$. We will prove the validity of (15). The proof for equation (14) is similar and can be found in [NRZ14, p. 53].

Since $h \equiv 0$ away from $(1-2\tau, 1-\tau)$, we have $L(s,t) = st$ for $s \in [0, 1-3\tau]$, so we need to check that

$$\int_{-x(s)}^{y(s)} (f(t)^2 - (st)^2)^{n-1}\, t\, dt = \int_{-x(s)}^{y(s)} (f_o(t)^2 - (st)^2)^{n-1}\, t\, dt \qquad \forall s \in [0, 1-3\tau].$$

Recall that $x = x_o$ and $y = y_o$ everywhere on this interval, so we can write x and y instead of x_o and y_o on the right-hand side.

Using the binomial formula, we see that it suffices to check that

$$\int_{-x(s)}^{y(s)} f(t)^{2j}\, t^{2(n-1-j)+1}\, dt = \int_{-x(s)}^{y(s)} f_o(t)^{2j}\, t^{2(n-1-j)+1}\, dt \tag{26}$$

for all $j = 1, \dots, n-1$ and $s \in [0, 1-3\tau]$. Since $f \equiv f_o$ outside $[-x(1-3\tau), y(1-3\tau)]$, splitting the integrals in (26) into three parts with ranges

$$[-x(s), -x(1-3\tau)], \quad [-x(1-3\tau), y(1-3\tau)], \quad [y(1-3\tau), y(s)],$$

it is enough to check (26) on the middle interval $[-x(1-3\tau), y(1-3\tau)]$.

To this end, we first take $s = 1-3\tau$, $k = n-2$ in (24) and conclude that

$$\int_{-x(1-3\tau)}^{y(1-3\tau)} f(t)^2\, t^{2n-3}\, dt = \int_{-x(1-3\tau)}^{y(1-3\tau)} f_o(t)^2\, t^{2n-3}\, dt, \tag{27}$$

which is (26) for $j = 1$ and $s = 1-3\tau$. Now we go "one step up," by taking $s = 1-3\tau$, $k = n-3$ in (24), to get

$$\int_{-x(1-3\tau)}^{y(1-3\tau)} (f(t)^2 - (st)^2)^2 t^{2n-5}\, dt = \int_{-x(1-3\tau)}^{y(1-3\tau)} (f_o(t)^2 - (st)^2)^2 t^{2n-5}\, dt.$$

The last equality together with (27) yield

$$\int_{-x(1-3\tau)}^{y(1-3\tau)} f(t)^4\, t^{2n-5}\, dt = \int_{-x(1-3\tau)}^{y(1-3\tau)} f_o(t)^4 t^{2n-5}\, dt,$$

which is (26) for $j = 2$ and $s = 1-3\tau$. Proceeding in a similar way we get (26) for $j = 1, \dots, n-1$ and $s = 1-3\tau$. This finishes the proof of Theorem 1 in even dimensions. $\square$

5. The case of odd $d \geq 3$

Note that $n = q + \frac{1}{2}$, $q \in \mathbb{N}$. Then (14), (15) take the form

$$\int_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{q+\frac{1}{2}} dt = \int_{-x_o(s)}^{y_o(s)} (f_o(t)^2 - L_o(s,t)^2)^{q+\frac{1}{2}} dt = \frac{\text{const}}{\sqrt{1+s^2}}, \tag{28}$$

$$\int_{-x(s)}^{y(s)} (f(t)^2 - L(s,t)^2)^{q-\frac{1}{2}} \frac{\partial L}{\partial s}(s,t)\, dt = 0, \tag{29}$$

where f_o, L_o, y_o, and x_o are defined by (13).

Our argument is similar to the one in [NRZ14, §4]. Our body of revolution K_f will be constructed as a perturbation of the Euclidean ball. We remark that in the case of odd dimensions, the perturbation will not be local, meaning that the resulting function $f(t)$ will be equal to $\sqrt{1-t^2}$ on $\left[-\frac{1}{\sqrt{1+(1-\tau)^2}}, \frac{1}{\sqrt{1+(1-\tau)^2}}\right]$ for some small $\tau > 0$.

We will make our construction in several steps corresponding to the slope ranges $s \in [1,\infty)$, $s \in [1-3\tau, 1]$, and $s \in (0, 1-3\tau]$. We will use different ways to describe the boundary of K_f within those ranges. We will *define* $f(t) = f_o(t)$ for $t \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. We will differentiate (28), (29) and rewrite the resulting equations in terms of x and y, to extend x and y to $[1-3\tau, 1]$ like we did in the even case. As before, f is related to x and y by (11). Finally, we will change the point of view and define the remaining part of f in terms of the functions $R(\alpha)$ and $r(\alpha)$, related to f by

$$f(R(\alpha)\cos\alpha) = R(\alpha)\sin\alpha, \ \ f(-r(\alpha)\cos\alpha) = r(\alpha)\sin\alpha, \ \ \alpha \in [0, \tfrac{\pi}{2}]. \tag{30}$$

Note that the *radial function* $\rho_K(w) = \sup\{t > 0 : tw \in K\}$ of the resulting body K satisfies

$$\rho_K(w) = \begin{cases} R(\alpha) & \text{if } w_1 > 0, \\ r(\alpha) & \text{if } w_1 < 0, \end{cases} \tag{31}$$

where $w = (w_1, \ldots, w_d) \in S^{d-1}$ and $\alpha \in [0, \frac{\pi}{2}]$, $\cos\alpha = |w_1|$.

Step 1. We put $x = x_o$, $y = y_o$ on $[1, \infty)$, which is equivalent to putting $f(t) = \sqrt{1-t^2}$ for $t \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

Step 2. Differentiating equation (28) $q+1$ times, we obtain

$$\begin{aligned}
&\Big(\frac{\partial}{\partial s}\Big)^{q+1}\int\limits_{-x(s)}^{y(s)}(f(t)^2-L(s,t)^2)^{q+\frac{1}{2}}\,dt\\
&=\Big(\int\limits_{-x(s)}^{-x_o(1)}+\int\limits_{y_o(1)}^{y(s)}\Big)\Big(\frac{\partial}{\partial s}\Big)^{q+1}\Big((f(t)^2-L(s,t)^2)^{q+\frac{1}{2}}\Big)\,dt\;+E_1(s) \qquad (32)\\
&=\Big(\frac{d}{ds}\Big)^{q+1}\frac{\text{const}}{\sqrt{1+s^2}},
\end{aligned}$$

where

$$E_1(s)=\int\limits_{-x_o(1)}^{y_o(1)}\Big(\frac{\partial}{\partial s}\Big)^{q+1}\Big((f_o(t)^2-L(s,t)^2)^{q+\frac{1}{2}}\Big)\,dt.$$

Note that, unlike the function Ξ_1 in the even-dimensional case, E_1 is well defined only for $s\le 1$ and only if $\|h\|_{C^1}$ is much smaller than 1. Also, even with these assumptions, $E_1(s)$ is C^∞ on $[0,1)$ but not at 1, where it is merely continuous.

Observe that

$$\Big(\frac{\partial}{\partial s}\Big)^{q+1}\Big((f(t)^2-L(s,t)^2)^{q+\frac{1}{2}}\Big)=\frac{J_1(s,t,f(t))}{\sqrt{f^2(t)-L^2(t)}},$$

where $J_1(s,t,f)$ is some polynomial expression in s, t, f, $h(s)$, and the derivatives of h at s.

Making the change of variables $t=-x(\sigma)$ in the integral $\int_{-x(s)}^{-x_o(1)}$, and $t=y(\sigma)$ in the integral $\int_{y_o(1)}^{y(s)}$ and using (11), we can rewrite the sum of the first two integrals in (32) as

$$\begin{aligned}
-\int\limits_{s}^{1}\Big[&\frac{J_1(s,-x(\sigma),L(\sigma,-x(\sigma)))}{\sqrt{L(\sigma,-x(\sigma))^2-L(s,-x(\sigma))^2}}\frac{dx}{ds}(\sigma)\\
&+\frac{J_1(s,y(\sigma),L(\sigma,y(\sigma)))}{\sqrt{L(\sigma,y(\sigma))^2-L(s,y(\sigma))^2}}\frac{dy}{ds}(\sigma)\Big]\,d\sigma.
\end{aligned}$$

Now write

$$L(\sigma,t)^2-L(s,t)^2=(L(\sigma,t)-L(s,t))(L(\sigma,t)+L(s,t))$$

and

$$L(\sigma,t)-L(s,t)=\sigma t+h(\sigma)-st-h(s)=(\sigma-s)(t+H(s,\sigma)),$$

where

$$H(s,\sigma) = \frac{h(\sigma) - h(s)}{\sigma - s} = \int_0^1 h'(s + (\sigma - s)\tau)\, d\tau$$

is an infinitely smooth function of s and σ. Let

$$K_1(s,\sigma,t) = \frac{J_1(s,t,L(\sigma,t))}{\sqrt{(t + H(s,\sigma))(L(\sigma,t) + L(s,t))}}.$$

The function K_1 is well defined and infinitely smooth for all s, σ, t satisfying $(t + H(s,\sigma))(L(\sigma,t) + L(s,t)) > 0$. If $\|h\|_{C^1}$ is small enough, this condition is fulfilled whenever s, $\sigma \in [\frac{1}{2}, 1]$ and $|t| > \frac{1}{2}$.

Now we can rewrite equation (32) in the form

$$-\int_s^1 \Big(K_1(s,\sigma,-x(\sigma))\,\frac{dx}{ds}(\sigma) \;+\; K_1(s,\sigma,y(\sigma))\,\frac{dy}{ds}(\sigma)\Big)\,\frac{d\sigma}{\sqrt{\sigma - s}} = -E_1(s) + \Big(\frac{d}{ds}\Big)^{q+1} \frac{\text{const}}{\sqrt{1+s^2}}. \tag{33}$$

Similarly, we can differentiate (29) q times and transform the resulting equation into

$$-\int_s^1 \Big(K_2(s,\sigma,-x(\sigma))\,\frac{dx}{ds}(\sigma) + K_2(s,\sigma,y(\sigma))\,\frac{dy}{ds}(\sigma)\Big)\,\frac{d\sigma}{\sqrt{\sigma - s}} = -E_2(s), \tag{34}$$

where K_2 is well defined and infinitely smooth in the same range as K_1. The function E_2 on the right-hand side of (34) is given by

$$E_2(s) = \int_{-x_o(1)}^{y_o(1)} \Big(\frac{\partial}{\partial s}\Big)^q \Big((f_o(t)^2 - L(s,t)^2)^{q-\frac{1}{2}}\,\frac{\partial L}{\partial s}(s,t)\Big)\, dt,$$

and everything that we said about E_1 applies to E_2 as well.

Equations (33) and (34) together can be written in the form

$$\int_s^1 \frac{K(s,\sigma,z(\sigma),\frac{dz}{ds}(\sigma))}{\sqrt{\sigma - s}}\, d\sigma = Q(s), \tag{35}$$

where, for $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z' = \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2$,

$$K(s,\sigma,z,z') = -\begin{pmatrix} K_1(s,\sigma,-x)\,x' \;+\; K_1(s,\sigma,y)\,y' \\ K_2(s,\sigma,-x)\,x' \;+\; K_2(s,\sigma,y)\,y' \end{pmatrix},$$

$$Q(s) = \begin{pmatrix} -E_1(s) + \left(\frac{d}{ds}\right)^{q+1} \frac{\text{const}}{\sqrt{1+s^2}} \\ -E_2(s) \end{pmatrix}.$$

By Lemma 8 in [NRZ14, p. 63] with $b = 1$, equation (35) is equivalent to

$$-\,G_2(s,s,z,z') + \int\limits_s^1 \frac{\partial}{\partial s} G_2(s,\sigma,z(\sigma),\frac{dz}{ds}(\sigma))\,d\sigma = \widetilde{Q}(s), \tag{36}$$

where

$$G_2(s,\sigma,z,z') = \int\limits_0^1 \frac{K(s+\tau(\sigma-s),\sigma,z,z')}{\sqrt{\tau(1-\tau)}}\,d\tau, \qquad \widetilde{Q}(s) = \frac{d}{ds}\int\limits_s^1 \frac{Q(s')}{\sqrt{s'-s}}\,ds'.$$

Note that

$$G_2(s,s,z,z') = C\,\cdot\,K(s,s,z,z'), \qquad C = \int\limits_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}}.$$

To reduce the resulting system of integro-differential equations to a pure system of integral equations we add two independent unknown functions x', y', let $z' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $z_o(s) = \begin{pmatrix} x_o(s) \\ y_o(s) \end{pmatrix}$, and add two new relations

$$z(s) = -\int\limits_s^1 z'(\sigma)\,d\sigma + z_o(1).$$

Together with (36), they lead to the system

$$\mathbf{G}(s,Z(s)) = \int\limits_s^1 \mathbf{\Theta}(s,\sigma,Z(\sigma))\,d\sigma + \mathbf{\Xi}(s), \qquad Z = \begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix}. \tag{37}$$

Here

$$\mathbf{G}(s,Z) = \begin{pmatrix} z \\ -G_2(s,s,z,z') \end{pmatrix}, \qquad \mathbf{\Theta}(s,\sigma,Z) = -\begin{pmatrix} z' \\ \frac{\partial}{\partial s} G_2(s,\sigma,z,z') \end{pmatrix},$$

and

$$\mathbf{\Xi}(s) = \begin{pmatrix} z_o(1) \\ \widetilde{Q}(s) \end{pmatrix}.$$

In what follows, we will choose h so that $\|h\|_{C^1}$ is much smaller than 1. In this case, $\mathbf{G}$, $\boldsymbol{\Theta}$ are well defined and infinitely smooth whenever s, $\sigma \in [\frac{1}{2}, 1]$, $|x|$, $|y| > \frac{1}{2}$, $z' \in \mathbb{R}^2$, and $\boldsymbol{\Xi}$ is well defined and infinitely smooth on $[\frac{1}{2}, 1)$. Observe also that

$$D_Z\mathbf{G}\Big|_{(s,Z(s))} = \begin{pmatrix} \mathbf{I} & 0 \\ * & \mathbf{A} \end{pmatrix},$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{A}(s,z) = C \cdot \mathbf{E}(s,z),$$

and

$$\mathbf{E}(s,z) = \begin{pmatrix} K_1(s,s,-x) & K_1(s,s,y) \\ K_2(s,s,-x) & K_2(s,s,y) \end{pmatrix}.$$

The function

$$Z_o(s) = \begin{pmatrix} z_o(s) \\ \frac{dz_o}{ds}(s) \end{pmatrix} = \begin{pmatrix} x_o(s) \\ y_o(s) \\ \frac{dx_o}{ds}(s) \\ \frac{dy_o}{ds}(s) \end{pmatrix}$$

solves the system (37) with $\mathbf{G}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$ corresponding to $h \equiv 0$ (we will denote them by $\mathbf{G}_o$, $\boldsymbol{\Theta}_o$, $\boldsymbol{\Xi}_o$) on $[\frac{1}{2}, 1]$, say.

We claim that

$$\det\left(D_Z\mathbf{G}_o\Big|_{(s,Z_o(s))}\right) = \det(\mathbf{A}_o(s, z_o(s))) \neq 0 \qquad \forall\ s \in [\tfrac{1}{2}, 1]. \tag{38}$$

Indeed, since $K_{1,2}(s,s,t)$ have the same signs as $J_{1,2}(s,\xi,L(s,t))$ and since

$$J_1(s,t,L(s,t)) = (2q+1)!!\left(-L(s,t)\,\frac{\partial L}{\partial s}(s,t)\right)^{q+1},$$
$$J_2(s,t,L(s,t)) = (2q-1)!!\left(-L(s,t)\frac{\partial L}{\partial s}(s,t)\right)^{q}\frac{\partial L}{\partial s}(s,t),$$

we conclude that the matrix $\mathbf{A}_o(s, z_o(s))$ has the same sign pattern as the matrix

$$\begin{pmatrix} (-1)^{q+1} & (-1)^{q+1} \\ (-1)^q(-x_o(s)) & (-1)^q y_o(s) \end{pmatrix};$$

i.e., the signs in the first row are the same, and the signs in the second one are opposite.

Thus, (38) follows. In particular,

$$\det\left(D_Z\mathbf{G}_o\Big|_{(1,Z_o(1))}\right) \neq 0.$$

Lemma 8 from [NRZ14, p. 63] then implies that we can choose some small $\tau > 0$ and construct a C^k-close to $Z_o(s)$ solution $Z(s)$ of (37) on $[1-3\tau, 1]$ whenever $\mathbf{G}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$ are sufficiently close to $\mathbf{G}_o$, $\boldsymbol{\Theta}_o$, $\boldsymbol{\Xi}_o$ in C^k on certain compact sets. Since $\mathbf{G}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$ and their derivatives are (integrals of) explicit elementary expressions in Z, s, σ, $h(s)$, and the derivatives of $h(s)$, this closeness condition will hold if h and sufficiently many of its derivatives are close enough to zero. Moreover, since h vanishes on $[1-\tau, 1]$, the assumptions of Remark 2 from [NRZ14, p. 66] are satisfied and we have $Z(s) = Z_o(s)$ on $[1-\tau, 1]$.

The x and y components of Z solve the equations obtained by differentiating (28) and (29). The passage to (28), (29) is now exactly the same as in the even case.

Step 3. From now on, we change the point of view and switch to the functions $R(\alpha)$ and $r(\alpha)$, $\alpha \in (0, \frac{\pi}{2})$, related to f by (30). The functions x and y, which we have already constructed, implicitly define C^∞-functions $R_h(\alpha)$ and $r_h(\alpha)$ for all α with $\tan\alpha > 1 - 3\tau$.

Instead of parametrizing hyperplanes by the slopes s of the corresponding linear functions, we will parametrize them by the angles β they make with the x_1-axis, where β is related to s by $\tan\beta = s$.

Our next task will be to derive the equations that will ensure that all *central* sections corresponding to angles β with $\tan\beta < 1 - 2\tau$ are the cutting sections with equal moments with respect to any $(d-2)$-dimensional subspace passing through the origin. We will also ensure that the origin is the center of mass of these sections. Note that the sections are already defined and satisfy these properties when $\tan\beta \in (1-3\tau, 1-2\tau)$.

It will be convenient to rewrite conditions (7), (8) and (9) in terms of the spherical Radon transform (see [Gar06, pp. 427–436]), defined as

$$\mathcal{R}f(\xi) = \int\limits_{S^{d-1}\cap\xi^\perp} f(w)\, dw, \qquad f \in C(S^{d-1}), \qquad \xi \in S^{d-1}.$$

We will use the following proposition.

Proposition 2. *Let K be a convex body of revolution about the x_1-axis containing the origin in its interior, and let $\xi = (\pm\sin\alpha, 0, \dots, 0, \mp\cos\alpha) \in S^{d-1}$ be the unit vector corresponding to the angle $\alpha \in [0, \frac{\pi}{2})$. Then the center of mass of the* central section $K \cap \xi^\perp$ *is at the origin if and only if*

$$(\mathcal{R}(w_j \rho_K^d(w))(\xi) = 0, \qquad j = 1, \dots, d-1. \tag{39}$$

Also, the moments of inertia of the central section $K \cap \xi^\perp$ with respect to any $(d-2)$-dimensional subspace Π are constant independent of Π if and only if

$$(\mathcal{R}(w_1^2 \rho_K^{d+1}(w))(\xi) = \mathrm{const}(d+1)(1-\xi_1^2), \tag{40}$$

$$(\mathcal{R}(w_j^2 \rho_K^{d+1}(w))(\xi) = \mathrm{const}(d+1) \quad \forall\, j = 2, \dots, d-1, \tag{41}$$

and

$$(\mathcal{R}(w_j w_l \rho_K^{d+1}(w))(\xi) = 0, \qquad j, l = 1, \dots, d-1, \quad j \neq l. \tag{42}$$

Proof. If the center of mass of $K \cap \xi^\perp$ is at the origin, we have

$$\frac{1}{\mathrm{vol}_{d-1}(K \cap \xi^\perp)} \int\limits_{K \cap \xi^\perp} x \, dx = 0.$$

Passing to polar coordinates in $\xi^\perp$ and taking into account the fact that for $w \in \xi^\perp$ we have $w_d = w_1 \tan\alpha$, we obtain the first statement of the lemma.

Let Π be any $(d-2)$-dimensional subspace of $\xi^\perp$, and let $u = u_{d-1}$ be a unit vector in $\xi^\perp$ orthogonal to Π. By (3) the condition on the moments reads as

$$(43) \qquad I_{K\cap\xi^\perp}(\Pi) = \int\limits_{K\cap\xi^\perp} (x \cdot u)^2 \, dx = \mathrm{const} \qquad \forall u \in S^{d-1} \cap \xi^\perp.$$

Denote by $\iota_1, \dots \iota_{d-1}$ the orthonormal basis in $\xi^\perp$ such that $\iota_1 = \cos\alpha \; e_1 + \sin\alpha \; e_d$ and $\iota_j = e_j$ for $j = 2, \dots, d-1$. Passing to polar coordinates and decomposing u in the basis $\{\iota_j\}_{j=1}^{d-1}$, we see that the moments of inertia of the central section $K \cap \xi^\perp$ with respect to any $(d-2)$-dimensional subspace are constant if and only if

$$(44) \qquad (\mathcal{R}((w \cdot \iota_1)^2 \rho_K^{d+1}(w))(\xi) = \mathrm{const}(d+1),$$

equation (41) holds, and

$$(45) \qquad (\mathcal{R}((w \cdot \iota_j)(w \cdot \iota_l)\rho_K^{d+1}(w))(\xi) = 0, \qquad j, l = 1, \dots, d-1, \quad j \neq l;$$

see the proof of Theorem 1 in [Rya20]. Since $w \cdot \iota_1 = w_1 \cos\alpha + w_d \sin\alpha$ and $w_d = w_1 \tan\alpha$, we see that (44) and (45) are equivalent to (40) and (42). This gives the second statement, and the lemma is proved. □

We remark that for any body of revolution around the x_1-axis, (39) holds for $j = 2, \dots, d-1$. Taking $u = \iota_j$ in the integral in (43), by rotation invariance we obtain that the moments in (41) are equal for $j = 2, \dots, d-1$. Also, arguing as at the end of the proof of Lemma 3 we see that (42) is valid.

By these remarks, Step 2, Lemma 4 with $s_o = 1 - 3\tau$ and Proposition 2 with $K = K_f$, when K_f is the body of revolution we are constructing, equations (39), (40), (41) and (42) hold if $\tan\alpha \in (1-3\tau, 1-2\tau)$ with the constants in (40), (41) independent of ξ. Also, the left-hand sides of (39), (40) and (41) are already defined on the cap

$$\mathcal{U}_\tau = \{\xi \in S^{d-1} : \xi_1 = \pm\sin\alpha, \quad \alpha \in [0, \frac{\pi}{2}], \quad \tan\alpha \geq 1 - 3\tau\}$$

and are smooth even rotation invariant functions there.

Assume for a moment that we have constructed a smooth body K_f so that conditions

$$(46) \quad (\mathcal{R}(w_1^2 \rho_{K_f}^{d+1}(w))(\xi) = \mathrm{const}(d+1)(1-\xi_1^2), \qquad (\mathcal{R}(w_1 \rho_{K_f}^{d}(w))(\xi) = 0$$

hold for all unit vectors $\xi \in S^{d-1}$ with $\xi_1 = \pm\sin\alpha$, corresponding to the angles $\alpha \in [0, \frac{\pi}{2}]$ such that $\tan\alpha < 1 - 2\tau$. Then by the above remarks, Proposition 2 and the converse part of Lemma 4 with $s_o = 0$, conditions (14), (15) of Proposition 1 are satisfied for all $s > 0$ and K_f floats in equilibrium in every orientation at the level $\frac{\mathrm{vol}_d(K)}{2}$.

Thus, it remains to construct the part of K_f so that (46) holds for all unit vectors ξ corresponding to the angles $\alpha \in [0, 1-2\tau]$. To this end, denote by φ_h and ψ_h the left-hand sides of (46) defined on $\mathcal{U}_\tau$. We put $\varphi_h(\xi) = \mathrm{const}(d+1)(1-\xi_1^2)$ and $\psi_h(\xi) = 0$ for $\xi \in S^{d-1}$ such that $\xi_1 = \pm\sin\alpha$ and $\tan\alpha \in [0, 1-2\tau]$. This definition agrees with the one we already have when $\tan\alpha \in [1-3\tau, 1-2\tau]$, so φ_h and ψ_h are even rotation invariant infinitely smooth functions on the entire sphere.

Recall that the values of $\mathcal{R}g(\xi)$ for all $\xi \in S^{d-1}$ such that $\xi_1 = \pm\sin\alpha$ and $\tan\alpha > 1-3\tau$ are completely determined by the values of the even function $g(w)$ for all $w \in S^{d-1}$ satisfying $w_1 = \pm\cos\alpha$ and $\tan\alpha > 1-3\tau$. Moreover, for *bodies of revolution* (but not in general) the converse is also true. (See the explicit inversion formula in [Gar06, p. 433, formula (C.17)].)

Since the equation $\mathcal{R}g = \widetilde{g}$ with even C^∞ right-hand side $\widetilde{g}$ is equivalent to

$$\frac{g(\xi) + g(-\xi)}{2} = \mathcal{R}^{-1}\widetilde{g}(\xi),$$

we can rewrite the equations in (46) as

$$w_1^2(\rho_K^{d+1}(w) + \rho_K^{d+1}(-w)) = 2(\mathcal{R}^{-1}\varphi_h)(w) \tag{47}$$

and

$$w_1(\rho_K^d(w) - \rho_K^d(-w)) = 2(\mathcal{R}^{-1}\psi_h)(w). \tag{48}$$

The already constructed part of ρ_K satisfies these equations for the vectors $w \in S^{d-1}$ such that $w_1 = \pm\cos\alpha$ and $\tan\alpha > 1-3\tau$.

Since the spherical Radon transform commutes with rotations and our initial ρ_K was rotation invariant, the even functions $2\mathcal{R}^{-1}\varphi_h(w)$, $2\mathcal{R}^{-1}\psi_h(w)$ are rotation invariant as well and can be written as $\Phi_h(\alpha)$ and $\Psi_h(\alpha)$, where $w \in S^{d-1}$ is such that $w_1 = \pm\cos\alpha$ and $\alpha \in [0, \frac{\pi}{2}]$. Note that the mappings $h \mapsto \Phi_h$, $h \mapsto \Psi_h$ are continuous from C^{k+d+1} to C^k, say. Thus, for all h sufficiently close to zero in C^{k+d+1}, Φ_h and Ψ_h will be close to $\Phi_0 \equiv 2w_1^2$ and $\Psi_0 \equiv 0$ in C^k.

We will be looking for a rotation invariant solution ρ_K of (47) and (48), which will be described in terms of the two functions $R(\alpha)$ and $r(\alpha)$ related to it by (31). Equations (47) and (48) translate into

$$R^{d+1}(\alpha) + r^{d+1}(\alpha) = \frac{\Phi_h(\alpha)}{\cos^2\alpha}, \quad R^d(\alpha) - r^d(\alpha) = \frac{\Psi_h(\alpha)}{\cos\alpha}. \tag{49}$$

Equations (49), together with the conditions $R(\alpha) > 0$ and $r(\alpha) > 0$, determine $R(\alpha)$ and $r(\alpha)$ uniquely, and they coincide with the functions R_h and r_h obtained in Step 2 for all $\alpha \in [0, \frac{\pi}{2}]$ with $\tan\alpha \geq 1-3\tau$. Therefore, any solution R, r of this system will satisfy $R(\alpha) = R_h(\alpha)$, $r(\alpha) = r_h(\alpha)$ in this range.

If h and several of its derivatives are small enough, the functions $\Phi_h - 2w_1^2$, Ψ_h and several of their derivatives are uniformly close to zero. Since the map $(R, r) \mapsto (R^{d+1} + r^{d+1}, R^d - r^d)$ is smoothly invertible near the point $(1, 1)$ by the inverse function theorem, the functions R, r exist in this case on the entire interval $[0, \frac{\pi}{2}]$, and are close to 1 in C^2. Moreover, $R'(0) = r'(0) = 0$, because $\Phi_h'(0) = 0$, $\Psi_h'(0) = 0$; otherwise the functions $\mathcal{R}^{-1}\varphi_h$, $\mathcal{R}^{-1}\psi_h$ would not be smooth at $(1, 0, \ldots, 0)$. This is enough to ensure that the body given by R and r is convex and corresponds to some strictly concave function f defined on $[-r(0), R(0)]$.

This completes the proof of Theorem 1 in the case of odd dimensions. $\square$

It remains to prove Theorem 2. Assume that a body $K \subset \mathbb{R}^3$ has density $\mathcal{D}$ and volume V. If K is submerged in liquid of density $\mathcal{D}'$ and V' is the volume of a submerged part, then, by Archimedes' law, $\mathcal{D}V = \mathcal{D}'V'$; cf. [Arc02, p. 257], [Zhu36, p. 657]. Taking $\mathcal{D}' = 1$ and $V' = \frac{1}{2}V$, we obtain the result. $\square$

Appendix A. Proof of Theorem 3 taken from [Olo41]

A.1. *The "if" part.* We begin with several auxiliary lemmas.

Lemma 5. *Let $d \geq 2$, let $M \subset \mathbb{R}^d$ be a convex body, and let $\varepsilon \in (0, 1)$. Consider the neighborhood of ∂M, $U_\varepsilon = U_\varepsilon(\partial M) = \{p \in \mathbb{R}^d : \operatorname{dist}(p, \partial M) < \varepsilon\}$. Then $\operatorname{vol}_d(U_\varepsilon) \leq 3\varepsilon S(M)$, provided ε is small enough.*

Proof. We fix a small $\varepsilon > 0$ (we will choose it precisely later) and claim first that

$$\operatorname{vol}_d(M \cap U_\varepsilon) \leq \operatorname{vol}_d((\mathbb{R}^d \setminus M) \cap U_\varepsilon). \tag{50}$$

Assume for a moment that M is a convex polytope, and consider the rectangular prisms T_F based on facets F of M of height 2ε, $T_F = F + [-\varepsilon v_F, \varepsilon v_F]$, where v_F is the outer unit normal vector to F such that $F + (0, \varepsilon v_F] \subset \mathbb{R}^d \setminus M$, $F + [-\varepsilon v_F, 0] \subset M$. The union of these prisms inside M contains $M \cap U_\varepsilon$, and the parts of prisms corresponding to the neighboring facets intersect. On the other hand, the parts outside of M do not intersect, and the inequality for polytopes follows from

$$\begin{aligned}\operatorname{vol}_d(M \cap U_\varepsilon) &\leq \operatorname{vol}_d\Big(\bigcup_F (F + [-\varepsilon v_F, 0])\Big) \\ &\leq \operatorname{vol}_d\Big(\bigcup_F (F + [0, \varepsilon v_F])\Big) \leq \operatorname{vol}_d((\mathbb{R}^d \setminus M) \cap U_\varepsilon).\end{aligned}$$

The general case can be obtained by approximation of M by polytopes and passing to the limit in the previous inequality. This proves the claim.

By (50) we have $\mathrm{vol}_d(U_\varepsilon) \le 2\mathrm{vol}_d((\mathbb{R}^d \setminus M) \cap U_\varepsilon)$, and it is enough to estimate the latter volume. To do this, we will use the fact that

$$(\mathbb{R}^d \setminus M) \cap U_\varepsilon \;\subseteq\; (M + \varepsilon B_2^d) \setminus M$$

and the definition of the surface area

$$S(M) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}_d(M + \varepsilon B_2^d) - \mathrm{vol}_d(M)}{\varepsilon}.$$

Taking ε_0 so small that for all $\varepsilon \in (0, \varepsilon_0)$ the above fraction is in $(\frac{S(M)}{2}, \frac{3S(M)}{2})$, we obtain the desired estimate

$$\mathrm{vol}_d(U_\varepsilon) \le 2(\mathrm{vol}_d(M + \varepsilon B_2^d) - \mathrm{vol}_d(M)) \le 3\varepsilon S(M). \qquad \square$$

To prove the next result we introduce some notation. Let P_H be the orthogonal projection onto a hyperplane H. For $\varepsilon > 0$, we let

$$\Xi_\varepsilon = P_{H(\xi)}(\{p \in \partial K : \mathrm{dist}(p, H(\xi)) < \varepsilon\}), \tag{51}$$

where $H(\xi)$ is a hyperplane for which (1) holds. Let D be the length of a diameter of K, and let

$$\mu = \frac{2D^d}{\mathrm{vol}_d(K \cap H^-(\xi))}. \tag{52}$$

We put

$$\Sigma_{\mu\varepsilon} = \{p \in H(\xi) : \mathrm{dist}(p, \partial K \cap H(\xi)) < \mu\varepsilon\}. \tag{53}$$

Lemma 6. *Let $\mathcal{E}$ be the maximal distance between $H(\xi)$ and any point in $K \cap H^-(\xi)$. Then $\Xi_\varepsilon \subset \Sigma_{\mu\varepsilon}$ for $\varepsilon \in (0, \mathcal{E})$ and $\mathrm{vol}_{d-1}(\Sigma_{\mu\varepsilon}) < 3c_d \mu D^{d-2}\varepsilon \to 0$ as $\varepsilon \to 0$.*

Proof. Consider a hyperplane $G(\xi) \in H^-(\xi)$ that is parallel to $H(\xi)$ and such that $\mathrm{dist}(H(\xi), G(\xi)) = \varepsilon$ for $\varepsilon \in (0, \mathcal{E})$. Consider also a hyperplane T containing any two corresponding parallel $(d-2)$-dimensional planes that support $K \cap H(\xi)$ and $K \cap G(\xi)$. In the half-space $H^-(\xi)$ containing these sections choose an angle γ between T and $H(\xi)$ that is not obtuse. (See Figure 4; cf. Figure 1 in [Olo41].)

Then

$$\mathcal{E} \le D \sin\gamma, \qquad \mathrm{vol}_d(K \cap H^-(\xi)) < D^{d-1}\mathcal{E} \le D^d \sin\gamma.$$

On the other hand, if $\lambda = \frac{\mathrm{vol}_d(K \cap H^-(\xi))}{\mathrm{vol}_d(K)}$, then

$$\mathrm{vol}_d(K \cap H^-(\xi)) \ge \frac{\lambda}{1+\lambda}\mathrm{vol}_d(K) \ge \frac{1}{2}\lambda\,\mathrm{vol}_d(K),$$

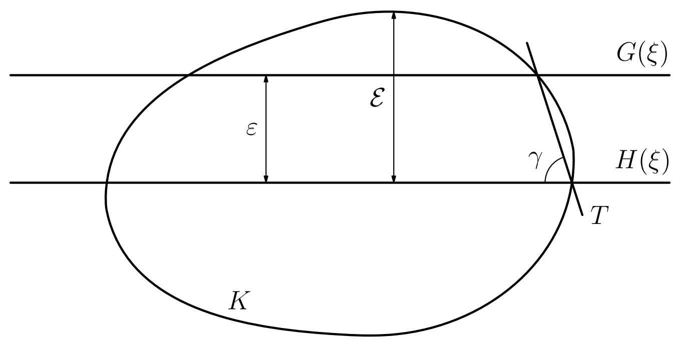

Figure 4. The hyperplanes $H(\xi)$, $G(\xi)$, and T.

which yields

$$\sin\gamma > \frac{\lambda \mathrm{vol}_d(K)}{2D^d}, \qquad |\cot\gamma| < \frac{2D^d}{\lambda \mathrm{vol}_d(K)} = \mu.$$

Since the distance between the corresponding $(d-2)$-dimensional support planes to $K \cap H(\xi)$ and $P_{H(\xi)}(K \cap G(\xi))$ is $\varepsilon|\cot\gamma| < \mu\varepsilon$, we see that Ξ_ε is a subset of $\Sigma_{\mu\varepsilon}$.

Identifying $H(\xi)$ with $\mathbb{R}^{d-1}$ and applying Lemma 5 with d and M replaced by $d-1$ and $K \cap H(\xi)$, respectively, we obtain

$$\mathrm{vol}_{d-1}(\Sigma_{\mu\varepsilon}) \le 3\mu\varepsilon S(K \cap H(\xi)) < 3\mu\varepsilon c_d D^{d-2} \to 0 \quad \text{as} \quad \varepsilon \to 0;$$

here the second inequality uses the estimate $S(K \cap H(\xi)) \le c_d D^{d-2}$, where c_d is a constant depending only on d. (See, for example, inequality (7) in [HCSSG04, Th. 1].) $\square$

Now consider a family $\mathcal{W} = \mathcal{W}_\Gamma$ of hyperplanes H satisfying (1) that are parallel to some $(d-2)$-dimensional subspace Γ. Each such hyperplane is determined by the angle $\theta \in [0, 2\pi]$ it makes with some fixed $H_0 \in \mathcal{W}$. (We take the orientation into account.) We will denote by $H(\theta)$ and $H(\theta + \Delta\theta)$ the hyperplanes in $\mathcal{W}$ making angles θ and $\theta + \Delta\theta$ with the chosen $H_0 = H(0) = H(2\pi)$.

Lemma 7. *For sufficiently small $\Delta\theta$, the $(d-2)$-dimensional plane $H(\theta) \cap H(\theta + \Delta\theta)$ passes through the interior of K.*

Proof. Without loss of generality we can assume that

$$\mathrm{vol}_d(K \cap H^-(\theta)) \le \mathrm{vol}_d(K \cap H^+(\theta))$$

and

$$\mathrm{vol}_d(K \cap H^-(\theta + \Delta\theta)) \le \mathrm{vol}_d(K \cap H^+(\theta + \Delta\theta)).$$

If the lemma is not true, then a priori we have the following three cases:

(1) one of $K \cap H^-(\theta)$ and $K \cap H^-(\theta + \Delta\theta)$ is strictly contained in the other;
(2) $K = (K \cap H^-(\theta)) \cup (K \cap H^-(\theta + \Delta\theta))$ but

$$\mathrm{vol}_d((K \cap H^-(\theta)) \cap (K \cap H^-(\theta + \Delta\theta))) > 0;$$

(3) $K \cap H^-(\theta)$ and $K \cap H^-(\theta + \Delta\theta)$ have no common point in the interior of $H^-(\theta)$.

The first two cases are impossible due to the fact that

$$\mathrm{vol}_d(K \cap H^-(\theta)) = \mathrm{vol}_d(K \cap H^-(\theta + \Delta\theta))$$

and

$$\mathrm{vol}_d(K \cap H^-(\theta)) \le \frac{1}{2}\mathrm{vol}_d(K), \quad \mathrm{vol}_d(K \cap H^-(\theta + \Delta\theta)) \le \frac{1}{2}\mathrm{vol}_d(K).$$

It remains to show that (3) is also impossible.

As argued in the first two cases, we can assume that $\delta \in (0, \frac{\mathrm{vol}_d(K)}{2})$. Let β be the smallest angle between $H(\theta)$ and the supporting hyperplanes to K at points in $\partial K \cap H(\theta)$. As in the proof of Lemma 6, one can show that

$$\beta > \sin\beta > \frac{\lambda \mathrm{vol}_d(K)}{2D^d} = \frac{1}{\mu},$$

where μ is defined by (52) with $H(\xi)$ replaced by $H(\theta)$.

Observe that one of the two supporting hyperplanes to K, which are parallel to $H(\theta + \Delta\theta)$, must also support $K \cap \mathrm{int}\, H^-(\theta)$, provided that $\Delta\theta \in (0, \frac{1}{\mu})$. Denote this hyperplane by $\widetilde{H}(\theta + \Delta\theta)$. We will show that

$$\Big(K \cap \mathrm{int}\, H^-(\theta) \cap \mathrm{int}\, H^-(\theta + \Delta\theta)\Big) \cap \widetilde{H}(\theta + \Delta\theta) \neq \emptyset \tag{54}$$

holds, which contradicts (3). To prove (54), we consider two more cases:

(3a) the part of K between $\widetilde{H}(\theta + \Delta\theta)$ and $H(\theta + \Delta\theta)$ strictly contains $K \cap H^-(\theta)$;
(3b) $K \cap H^-(\theta + \Delta\theta) \subsetneq K \cap H^+(\theta)$ for all $\Delta\theta \in (0, \frac{1}{\mu})$.

However, the case (3a) is similar to (1), hence it cannot occur. The case (3b) is also impossible. Otherwise, we would have

$$\delta = \lim_{\Delta\theta \to 0} \mathrm{vol}_d(K \cap H^-(\theta + \Delta\theta)) = \mathrm{vol}_d(K \cap H^+(\theta)) = \mathrm{vol}_d(K) - \delta > \frac{\mathrm{vol}_d(K)}{2},$$

which contradicts our choice of δ. This shows that (54) holds, and this finishes the proof of the lemma. □

Now choose a "moving" system of coordinates in which the $(d-2)$-dimensional plane $H(\theta) \cap H(\theta + \Delta\theta)$ is the $p_1 p_2 \cdots p_{d-2}$-coordinate plane. Since in our argument $\Delta\theta$ will tend to zero, we assume that $\Delta\theta$ is acute and $\Delta\theta < \frac{1}{\mu}$. Therefore, by the previous lemma, $H(\theta) \cap H(\theta + \Delta\theta)$ intersects the interior

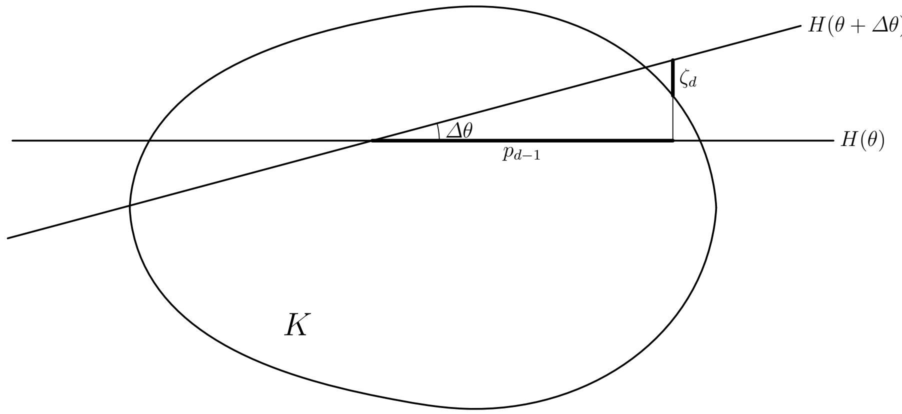

Figure 5. The function ζ_d.

of K. We also assume that the axis $p_{d-1} = p_{d-1}(\theta, \Delta\theta)$ is in $H(\theta)$ and the axis $p_d = p_d(\theta, \Delta\theta)$ is orthogonal to $H(\theta)$.

Our next goal is to write a formula for

$$\Delta V = \mathrm{vol}_d(K \cap H^-(\theta)) - \mathrm{vol}_d(K \cap H^-(\theta + \Delta\theta)) \tag{55}$$

in terms of $\int\limits_{K\cap H(\theta)} p_{d-1} \tan \Delta\theta \, dp$. To do this, we let

$$\Lambda = (K \cap H(\theta)) \triangle P_{H(\theta)}(K \cap H(\theta + \Delta\theta))$$

and we estimate the error $\zeta_d = \zeta_d(\theta, \Delta\theta)$ of $p_d = p_{d-1} \tan \Delta\theta$ in Λ that is obtained during the computation of ΔV using the latter integral (see Figure 5).

More precisely, ζ_d is defined as follows. Let $p \in \Lambda$, let l be the line parallel to the p_d-axis passing through p, and let $\zeta^+ = l \cap H(\theta + \Delta\theta)$. We put $\zeta^- = l \cap \partial K \cap H^-(\theta + \Delta\theta) \cap H^+(\theta)$ or $\zeta^- = l \cap \partial K \cap H^+(\theta + \Delta\theta) \cap H^-(\theta)$, provided $\zeta_d^+ > 0$ or $\zeta_d^+ < 0$ correspondingly. We have $\zeta_d^+ = p_{d-1} \tan \Delta\theta$. If $\zeta_d^+ > 0$, then $[\zeta^-, \zeta^+] \subset K$ or $[p, \zeta^-] \subset K$, and we put $\zeta_d = \zeta_d^-$ or $\zeta_d = \zeta_d^+ - \zeta_d^-$ correspondingly. If $\zeta_d^+ < 0$, then $[\zeta^+, \zeta^-] \subset K$ or $[\zeta^-, p] \subset K$, and we define $\zeta_d = \zeta_d^-$ or $\zeta_d = \zeta_d^+ - \zeta_d^-$.

The next lemma is a direct consequence of the fact that all hyperplanes in $\mathcal{W}$ satisfy (1).

Lemma 8. *Let ΔV be defined by* (55). *Then*

$$\Delta V = \int\limits_{K\cap H(\theta)} p_{d-1} \tan \Delta\theta \, dp - \int\limits_{\Lambda} \zeta_d \, dp = 0. \tag{56}$$

We are ready to finish the proof of the "if" part of Theorem 3. Let $p_{d-1}(\mathcal{C}(K \cap H(\theta)))$ be the $(d-1)$-coordinate of $\mathcal{C}(K \cap H(\theta))$ with respect to the moving coordinate system. By (56), we have

$$p_{d-1}(\mathcal{C}(K\cap H(\theta))) = \frac{\int\limits_{K\cap H(\theta)} p_{d-1}\ \ dp}{\mathrm{vol}_{d-1}(K\cap H(\theta))} = \frac{\int\limits_{\Lambda} \zeta_d\ \ dp}{\mathrm{vol}_{d-1}(K\cap H(\theta))\tan\varDelta\theta}.$$

Let $\Xi_{D\sin\varDelta\theta}$ be defined as in (51) with $H(\xi)$ replaced by $H(\theta)$. Since for every $p\in\Lambda$ there exists $q\in\partial K$ such that $P_{H(\theta)}q = p$ and $\mathrm{dist}(q, H(\theta)) < D\sin\theta$, we see that $\Lambda\subset\Xi_{D\sin\varDelta\theta}$. Applying Lemma 6, we have $\Xi_{D\sin\varDelta\theta}\subset\Sigma_{\mu D\sin\varDelta\theta}$ and

$$\mathrm{vol}_{d-1}(\Lambda)\le\mathrm{vol}_{d-1}(\Xi_{D\sin\varDelta\theta})\le\mathrm{vol}_{d-1}(\Sigma_{\mu D\sin\varDelta\theta})\le 3c_d\mu D^{d-1}\varDelta\theta\to 0$$

as $\varDelta\theta\to 0$. Using the estimate $|\zeta_d|\le D\tan\varDelta\theta$, this gives

$$|p_{d-1}(\mathcal{C}(K\cap H(\theta)))|\le\frac{D\tan\varDelta\theta\ \mathrm{vol}_{d-1}(\Lambda)}{\mathrm{vol}_{d-1}(K\cap H(\theta))\tan\varDelta\theta}\to 0$$

as $\varDelta\theta\to 0$. Therefore, as $\varDelta\theta\to 0$, the $(d-2)$-dimensional plane $H(\theta)\cap H(\theta+\varDelta\theta)$ tends to the limiting position $\Pi_\Gamma(\theta)$ that passes through the center of mass of $K\cap H(\theta)$ and which is parallel to Γ.

To show that $\mathcal{C}(K\cap H(\theta))$ is the characteristic point of the family of cutting hyperplanes with respect to $H(\theta)$, we consider any $(d-2)$-dimensional subspace $\overline{\Gamma}$ that is parallel to $H(\theta)$ and repeat the above considerations for the family of cutting hyperplanes $\mathcal{W}_{\overline{\Gamma}}$ that are parallel to $\overline{\Gamma}$. We see that if the plane $\overline{H}(\theta+\varDelta\theta)\in\mathcal{W}_{\overline{\Gamma}}$ tends to $H(\theta)$ as $\varDelta\theta\to 0$, then $\mathcal{C}(K\cap H(\theta))\in\Pi_{\overline{\Gamma}}(\theta)$, where $\Pi_{\overline{\Gamma}}(\theta)$ is the corresponding limiting position of $H(\theta)\cap\overline{H}(\theta+\varDelta\theta)$ as $\varDelta\theta\to 0$. Therefore, $\mathcal{C}(K\cap H(\theta))\in\bigcap\Pi_{\overline{\Gamma}}(\theta)$, where the intersection is taken over all $(d-2)$-dimensional subspaces $\overline{\Gamma}$ parallel to $H(\theta)$. This shows that the characteristic point of the family of cutting hyperplanes with respect to $H(\theta)$ is $\mathcal{C}(K\cap H(\theta))$.

Since the subspace Γ and the angle θ were chosen arbitrarily, we obtain the proof of the "if" part of the theorem.

A.2. *Proof of the converse part of Theorem* 3. Let $\mathcal{Q}$ be the family of hyperplanes satisfying the hypotheses of the converse part of the theorem. Let also Γ be an arbitrary $(d-2)$-dimensional subspace, and let $\mathcal{V}\subset\mathcal{Q}$ be the family of hyperplanes H parallel to Γ and such that the centers of mass of $K\cap H$, $H\in\mathcal{V}$, coincide with the characteristic points of $\mathcal{Q}$ with respect to H. Also, as above, choose an arbitrary angle θ, the hyperplanes $H(\theta)$ and $H(\theta+\varDelta\theta)$ in $\mathcal{V}$ and a "moving" coordinate system. Since $\mathcal{C}(K\cap H(\theta))$ is the characteristic point of $\mathcal{Q}$ with respect to $H(\theta)$, we can assume that $p_{d-1}(\mathcal{C}(K\cap H(\theta)))$, as a function of $\varDelta\theta$, tends to zero as $\varDelta\theta\to 0$.

Using (56) we have

$$\frac{\Delta V}{\Delta\theta} = \frac{\tan\Delta\theta}{\Delta\theta}\int\limits_{K\cap H(\theta)} p_{d-1}\,dp - \int\limits_{\Lambda}\frac{\zeta_d}{\Delta\theta}\,dp. \tag{57}$$

Since $\mathcal{C}(K\cap H(\theta+\Delta\theta))\to\mathcal{C}(K\cap H(\theta))$ and $\partial K\cap H(\theta+\Delta\theta)\to\partial K\cap H(\theta)$ as $\Delta\theta\to 0$, the set Λ defined in Lemma 8 satisfies $\mathrm{vol}_{d-1}(\Lambda)\to 0$ as $\Delta\theta\to 0$. Using this and the fact that $|\zeta_d|\le D\tan\Delta\theta$, we see that both summands in the right-hand side of the above identity tend to 0 as $\Delta\theta\to 0$. This gives $\lim_{\Delta\theta\to 0}\frac{\Delta V}{\Delta\theta}=0$.

Consider the function $\xi\mapsto g(\xi):=\mathrm{vol}_d(K\cap H^-(\xi))$ on S^{d-1}, where $H(\xi)\in\mathcal{Q}$. We will show that g is identically constant on S^{d-1}. It is enough to prove that for every $\varsigma\in S^{d-1}\cap e_d^{\perp}$, g is constant on the meridian $M(\varsigma)=\{\xi=(\sin\varphi\,\varsigma,\cos\varphi)\in S^{d-1}:\varphi\in[0,\pi]\}$.

To this end, let $\varsigma\in S^{d-1}\cap e_d^{\perp}$ be fixed. Also, fix any $\xi=(\sin\varphi\,\varsigma,\cos\varphi)\in M(\varsigma)$ such that $\varphi\in(0,\pi)$. Since Γ and θ are at our disposal, we make the choice as follows. We take Γ to be the $(d-2)$-dimensional subspace orthogonal to the 2-dimensional subspace $\Gamma^{\perp}$ containing $M(\varsigma)$. Next, as above, we consider the corresponding family $\mathcal{V}$ of hyperplanes H parallel to Γ, and we choose $\theta=\theta(\xi)\in[0,2\pi)$ so that $H(\theta)$ has the outer normal vector ξ. Then $\frac{\partial g}{\partial\varphi}(\xi)=\lim_{\Delta\theta\to 0}\frac{\Delta V}{\Delta\theta}$, where $\frac{\Delta V}{\Delta\theta}$ is as in (57). We have proved above that the latter limit is zero, hence, $\frac{\partial g}{\partial\varphi}(\xi)=0$. Since the point $\xi\in M(\varsigma)\setminus\{\pm e_d\}$ was chosen arbitrarily, by the mean value theorem, we see that g is constant on $M(\varsigma)\setminus\{\pm e_d\}$. Since g is continuous on S^{d-1}, we conclude that it must be constant on $M(\varsigma)$ and on S^{d-1}. The proof of the converse part is complete.

This finishes the proof of Theorem 3. $\square$

Appendix B. Proof of the converse part of Theorem 4

Let Γ be any $(d-2)$-dimensional subspace of $\mathbb{R}^d$. We let the family $\mathcal{W}=\mathcal{W}_\Gamma$ of hyperplanes $H(\theta)$, $\theta\in[0,2\pi]$, satisfying (1) and that are parallel to Γ be as in the previous section. We will use the notation $\mathcal{C}(\theta)\in\mathcal{S}$ for the centers of mass of the corresponding "submerged" parts $K\cap H^-(\theta)$.

We remark that by [HSW19, Th. 1.2] the surface of centers $\mathcal{S}$ is C^1-smooth, and we will denote by $\mathcal{H}(\theta)$ the tangent hyperplane to $\mathcal{S}$ at $\mathcal{C}(\theta)$.

The following auxiliary result is well known (cf. [dLVP25, pp. 275–279] and [Zhu36, pp. 658–660]). Since these references are not readily available, we provide the proof for the reader's convenience.

Theorem 5. *Let $d\ge 2$, let $K\subset\mathbb{R}^d$ be a convex body, and let $\delta\in(0,\mathrm{vol}_d(K))$. Then for any Γ and for any $H(\theta)\in\mathcal{W}_\Gamma$, $\theta\in[0,2\pi]$, $\mathcal{H}(\theta)$ is parallel to $H(\theta)$. Also, the bounded set $L=L(\mathcal{S})$ with boundary $\mathcal{S}$ is a strictly convex body.*

Proof. Fix Γ and $\theta \in [0, 2\pi)$. Let $\mathcal{H}^{\pm}(\theta)$ be the corresponding half-spaces. We claim that $\mathcal{S} \subset \mathcal{H}^{+}(\theta)$.

Rotating and translating if necessary, we can assume that $H(\theta) = e_d^{\perp}$ and $K \cap H^{-}(\theta) \subset \{p \in \mathbb{R}^d : p_d \leq 0\}$. Let $H(\widetilde{\theta}) \in \mathcal{W}_{\Gamma}$, $\widetilde{\theta} \neq \theta$, $\widetilde{\theta} \in [0, 2\pi)$. To prove the claim, it is enough to show that $\mathcal{C}(\widetilde{\theta})$ is "above" $\mathcal{C}(\theta)$, i.e., $p_d(\mathcal{C}(\theta)) < p_d(\mathcal{C}(\widetilde{\theta}))$. Since $p_d > 0$ for all $p \in (K \cap H^{-}(\widetilde{\theta})) \setminus (K \cap H^{-}(\theta))$ but $p_d \leq 0$ for all $p \in (K \cap H^{-}(\theta)) \setminus (K \cap H^{-}(\widetilde{\theta}))$, we have

$$\begin{aligned} p_d(\mathcal{C}(\theta)) &= \frac{1}{\delta}\Big(\int\limits_{(K\cap H^{-}(\theta))\setminus(K\cap H^{-}(\widetilde{\theta}))} p_d\, dp + \int\limits_{K\cap H^{-}(\theta)\cap H^{-}(\widetilde{\theta})} p_d\, dp \Big) \\ &< \frac{1}{\delta}\Big(\int\limits_{(K\cap H^{-}(\widetilde{\theta}))\setminus(K\cap H^{-}(\theta))} p_d\, dp + \int\limits_{K\cap H^{-}(\theta)\cap H^{-}(\widetilde{\theta})} p_d\, dp \Big) = p_d(\mathcal{C}(\widetilde{\theta})) \end{aligned}$$

and the claim is proved. Since $\mathcal{S} \cap \mathcal{H}(\theta) = \mathcal{C}(\theta)$, the hyperplane $\mathcal{H}(\theta)$ is tangent to $\mathcal{S}$ at $\mathcal{C}(\theta)$.

Thus, for any $\xi \in S^{d-1}$, we have $\mathcal{S} \subset \mathcal{H}^{+}(\xi)$, $\mathcal{S} \cap \mathcal{H}(\xi) = \mathcal{C}_{\delta}(\xi)$ and $\min_{\{\xi \in S^{d-1}\}} |\mathcal{C}(K) - \mathcal{C}_{\delta}(\xi)| > 0$. We conclude that $L(\mathcal{S}) = \bigcap_{\{\xi \in S^{d-1}\}} \mathcal{H}^{+}(\xi)$ is a strictly convex body. □

To prove the converse part of Theorem 4 it is enough to show that the orthogonal projection of $\mathcal{S}$ onto any 2-dimensional subspace of $\mathbb{R}^d$ is a disc. Indeed, by applying [Gar06, Cor. 3.1.6, p. 101] to $L(\mathcal{S})$, we obtain that in this case $\mathcal{S}$ is a sphere. Using Theorem 5, as well as the fact that all normal lines of the sphere intersect at its center, we see that for every $\xi \in S^{d-1}$, the lines $\ell(\xi)$ passing through $\mathcal{C}(K) = \mathcal{C}(\mathcal{S})$ and $\mathcal{C}_{\delta}(\xi)$ are orthogonal to $H(\xi)$. By Definition 1 this means that K floats in equilibrium in every orientation.

Let Γ be as above, let $\Gamma^{\perp}$ be the 2-dimensional subspace orthogonal to Γ, and let $P = P_{\Gamma^{\perp}}$ be the orthogonal projection onto $\Gamma^{\perp}$. Let $\beta \subset \mathcal{S}$ be the shadow boundary of L with respect to $\Gamma^{\perp}$, i.e.,

$$\beta = \bigcup_{\{\theta \in [0, 2\pi]\}} L \cap \mathcal{H}(\theta) = \{\mathcal{C}_{\delta}(\theta) : \theta \in [0, 2\pi]\}.$$

To show that $P(L)$ is a disc for every Γ, we will prove the following lemma.

LEMMA 9. *Let $\xi(\theta) \in S^{d-1}$ be the outer normal vector to $H(\theta)$, and let $P\beta = \{P\mathcal{C}_{\delta}(\theta) : \theta \in [0, 2\pi]\}$ be parametrized as $\theta \mapsto \varrho(\theta)$, $\theta \in [0, 2\pi]$. Then*

$$\varrho'(\theta) = -\frac{1}{\delta} I_{K \cap H(\theta)}(\Pi)\ \xi'(\theta) \qquad \forall \theta \in [0, 2\pi], \tag{58}$$

where Π is the $(d-2)$-dimensional plane passing through $\mathcal{C}(K \cap H(\theta))$ and parallel to Γ.

Assume for a moment that (58) is proved. By conditions of the theorem, $I_{K\cap H(\theta)}(\Pi)$ is a constant c independent of Π and θ. Integrating both sides in (58) we have $\varrho(\theta) = -\frac{c}{\delta}\,\xi(\theta) + C$, where C is a constant vector. Hence, $P\beta$ is a circle. Since Γ was chosen arbitrarily, the projection of $\mathcal{S}$ onto any 2-dimensional subspace is a disc.

To finish the proof, it remains to prove the lemma.

Proof. We can assume that $H(\theta) = e_d^{\perp}$, $K \cap H^-(\theta) \subset \{p \in \mathbb{R}^d : p_d \le 0\}$ and $\varrho(\theta)$, $\xi(\theta)$, $\xi'(\theta)$ are 2-dimensional, i.e., $\varrho(\theta) = (\varrho_{d-1}(\theta), \varrho_d(\theta))$, $\xi(\theta) = (0,1)$, $\xi'(\theta) = (-1,0)$. Since the tangent vector $\varrho'(\theta)$ is parallel to $\mathcal{H}(\theta)$ and since $\mathcal{H}(\theta)$ is parallel to $H(\theta)$ by the previous theorem, we conclude that $\varrho_d'(\theta) = 0$.

To compute $\varrho_{d-1}'(\theta)$, we will estimate $\varrho_{d-1}(\theta+\mathit{\Delta}\theta) - \varrho_{d-1}(\theta)$ for $\mathit{\Delta}\theta$ small enough. As in the previous appendix, we choose a "moving" system of coordinates in which the $(d-2)$-dimensional plane $H(\theta) \cap H(\theta+\mathit{\Delta}\theta)$ is the $p_1p_2\cdots p_{d-2}$-coordinate plane. We have

$$\begin{aligned}\varrho_{d-1}(\theta+\mathit{\Delta}\theta) - \varrho_{d-1}(\theta) &= \frac{1}{\delta}\Big(\int\limits_{K\cap H^-(\theta+\mathit{\Delta}\theta)} p_{d-1}\,dp - \int\limits_{K\cap H^-(\theta)} p_{d-1}\,dp\Big)\\ &= \frac{1}{\delta}\Big(\int\limits_{K\cap H(\theta)} p_{d-1}^2 \tan \mathit{\Delta}\theta\,dp - \int\limits_{\Lambda} p_{d-1}\zeta_d\,dp\Big),\end{aligned}$$

where the last equality is similar to (56), and Λ and ζ_d are as in Lemma 8 (see Figure 5). Dividing both sides by $\mathit{\Delta}\theta$, passing to the limit as $\mathit{\Delta}\theta \to 0$ and using the "if" part of the theorem proved in the previous appendix, we obtain

$$\varrho_{d-1}'(\theta) = \frac{1}{\delta}\int\limits_{K\cap H(\theta)} p_{d-1}^2\,dp = \frac{1}{\delta} I_{K\cap H(\theta)}(\Pi).$$

This gives (58). □

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