Complex Analysis, Spring 2011.

Instructor: Dmitry Ryabogin

Assignment I.

1. **Problem 1.**

   a) When does $az + b\bar{z} + c = 0$ represent a line?

   b) Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

   c) Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z| = 1$ at right angles.

   Hint: Observe that $|a| = |1/\bar{a}|$, and $arg\, a = arg(1/\bar{a})$. Then use your knowledge from the high school.

2. **Problem 2.**

   a) Let $d(z, z')$ be the distance between the stereographic projections of z and z', and let Z and Z' denote the stereographic projections of z, z', and let N be a north pole. Show that the triangles NZZ' and Nzz' are similar, and use it to derive

   $$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}.$$

   b) Check that d is a **metric** on $\mathbf{C}$.

   c) What are stereographic projections of $1, -1, i, (1 - i)/\sqrt{2}$?

   d) Describe the stereographic projection of the pair of points that are symmetric with respect to: $z = 0$, real axis, unit circle (see Problem 1 c)).

3. **Problem 3.**

   a) Let the points $z_1, z_2, ...z_n$ be located in the half-plane, with the boundary line passing through the origin. Prove that the points $1/z_1, 1/z_2, ..., 1/z_n$ have the same property (with respect to what line?).

   b) Prove that

   $$z_1 + z_2 + ... + z_n \neq 0, \qquad \frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n} \neq 0.$$

4. **Problem 4.**

   a) The functions

   $$\frac{Rez}{z}, \qquad \frac{z}{|z|}, \qquad \frac{Re(z^2)}{|z|^2}, \qquad \frac{zRez}{|z|}$$

   are defined for $z \neq 0$. Is it possible to define them at 0 in such a way that they would be continuous at this point?

b) Are functions

$$\frac{1}{1-|z|}, \qquad \frac{1}{1+|z|^2}$$

continuous in $(|z|<1)$? Are they uniformly continuous there?

c) Find constants a, b, c for which the function $f(z) = x + ay + i(bx + cy)$ is analytic.

d) Let $z = r(\cos\phi + i\sin\phi)$ and $f(z) = u(r,\phi) + iv(r,\phi)$. Write down the Cauchy-Riemann equations in polar coordinates.

e) Prove that $w = zRez$ is differentiable only at $z = 0$ and find $w'(0)$.